GROWING SIMPLICIAL COMPLEXES AND EMERGENT NETWORK GEOMETRY

Ginestra Bianconi

School of Mathematical Sciences, Queen Mary University of London, London, UK
Simplicial complexes are characterizing the interaction between two or more nodes and are formed by nodes, links, triangles, tetrahedra etc.

$d=2$ simplicial complex

$d=3$ simplicial complex
A simplex of dimension $d$ is a set of $d+1$ nodes

- it indicates the interactions between the nodes
- it admits a geometrical interpretation
Faces of a simplex

A face of a $d$-dimensional simplex is a $\delta$-dimensional simplex (with $\delta<d$) formed by a non-empty subset of its nodes.
A simplicial complex $K$ is a set of simplices with the following condition:

If a simplex $\mu$ belongs to $K$ every face of $\mu$ must belong to $K$.
From every network we can extract a simplicial complex called **clique complex** by associating a simplex to every clique (fully connected subgraph).
Clique percolation reveals overlapping communities by characterizing how cliques percolate in networks.

World association network

Palla et al.
Nature (2005)
From a simplicial complex to a network

From every simplicial complex we can extract a network by considering exclusively nodes and links
Brain data as simplicial complexes

Giusti et al (2016)
Protein interaction networks as simplicial complexes

- **Nodes**: proteins
- **Simplices**: protein complexes

Wan et al. Nature 2015
Collaboration networks as simplicial complexes

Actor collaboration networks

• **Nodes**: Actors
• **Simplicies**: Co-actors of a movie

Scientific collaboration networks

• **Nodes**: Scientists
• **Simplicies**: Co-authors
Quantum Spacetime

THE FABRIC OF REALITY

One clue
Quantum effects in the gravitational field of a black hole cause it to radiate energy as if it were hot, implying a deep connection between quantum theory, gravity and thermodynamics - the science of heat.

Singularity
The black hole's mass is concentrated at a singularity of infinite curvature.

3. Causal sets
The building blocks of spacetime are point-like "events" that form an ever-expanding network linked by causality.

An earlier event can affect a later one, but not vice versa.

4. Causal dynamical triangulations
Computer simulations approximate the fundamental quantum reality as tiny polygonal shapes, which obey quantum rules as they spontaneously self-assemble into larger patches of space-time.

Space at an instant
Space-time

5. Holography
A three-dimensional (3D) universe contains black holes and strings governed solely by gravity, whereas its 2D boundary contains ordinary particles governed solely by standard quantum-field theory.

Anything happening in the 3D interior can be described as a process on the 2D boundary; and vice versa.

1. Gravity as thermodynamics
The equations of gravity can actually be derived from thermodynamics, without reference to space-time curvature.

This suggests that gravity on a macroscopic scale is just an average of the behaviour of some still-unknown "atoms" of space-time.

2. Loop quantum gravity
The Universe is a network of intersecting quantum threads, each of which carries quantum information about the size and shape of nearby space.

Imagine drawing a closed surface anywhere in the network; its volume is determined by the intersections it encloses, its area by the number of threads that pierce it.
Quantum gravity approaches networks and simplicial complexes

- Causal-Dynamical-Triangulations
- Tensor networks
- Group Field Theory
- Causal sets
- Loop Quantum Gravity
- Spin-Foams
- ...

•
Spatial networks: infrastructures

Spatial networks: models and numerical methods

Random Geometric Networks

Tesselations

Spatial networks: Apollonian network

- The Apollonian gasket from Apollonius of Perga c.262-c.190 B.C. is formed by triple of circle each one tangent to the other two.
- The Apollonian network is generated by placing a node to each center of the circles and connecting nodes corresponding to tangent circles.

The Apollonian network is scale-free with exponent \( \gamma = 1 + \frac{\ln 3}{\ln 2} \approx 2.585 \)
Scale-free networks

- Technological networks
  - Internet
  - World-Wide Web
- Biological networks
  - Metabolic networks,
  - Protein-interaction networks,
  - Transcription networks
- Transportation networks
  - Airport networks
- Social networks
  - Collaboration networks
  - Citation networks
  - Facebook
- Economical networks
  - Networks of shareholders
  - The World Trade Web
Growth by uniform attachment of links

(1) **GROWTH**: At every timestep we add a new node with $m$ edges (connected to the nodes already present in the system).

(2) **UNIFORM ATTACHMENT**: The probability $\Pi_i$ that a new node will be connected to node $i$ is uniform.

$$\Pi_i = \frac{1}{N}$$

(1) **GROWTH**: At every timestep we add a new node with $m$ edges (connected to the nodes already present in the system).

(2) **PREFERENTIAL ATTACHMENT**: The probability $\Pi(k_i)$ that a new node will be connected to node $i$ depends on the connectivity $k_i$ of that node.

\[ \Pi(k_i) = \frac{k_i}{\sum_j k_j} \]

The Bianconi-Barabasi model

Growth:

– At each time a new node and $m$ links are added to the network.
– To each node $i$ we assign an energy $\varepsilon_i$ from a $g(\varepsilon)$ distribution

Preferential attachment towards high degree low energy nodes:

– Each node connects to the rest of the network by $m$ links attached preferentially to well connected, low energy nodes.

\[ \Pi_i = \frac{e^{-\beta \varepsilon_i} k_i}{\sum_j e^{-\beta \varepsilon_j} k_j} \]
Bose-Einstein condensation in complex networks

Scale-Free Fit-get-rich Phase

$\beta < \beta_C$

Bose-Einstein condensate Phase

$\beta > \beta_C$

G. Bianconi, A.-L. Barabási 2001
It is believed that most complex networks have an hidden metric such that the nodes close in the hidden metric are more likely to be linked to each other.
Hyperbolic geometry of complex networks could contribute to improve routing algorithms.
Emergent geometry

In the framework of emergent geometry, networks with hidden geometry are generated by equilibrium or non-equilibrium dynamics that makes no use of the hidden geometry.
Saturated and Unsaturated links

- $\rho_{ij} = 1$ if the link is unsaturated, i.e., less than $m$ triangles are incident on it
- $\rho_{ij} = 0$ if the link is saturated, i.e., the number of incident triangles is given by $m$
Growing Simplicial Complex

Growing Geometrical Network

Process (a)

We choose a link \((i,j)\) with probability

and glue a new triangle the link

\[
\Pi_{i,j}^{[1]} = \frac{a_{ij} \rho_{ij}}{\sum_{r,s} a_{rs} \rho_{rs}}
\]
We choose a two adjacent unsaturated links and we add the link between the nodes at distance 2 and all triangles that this link closes as long that this is allowed.
The model

Starting from an initial triangle,
At each time

• process (a) takes place and

• process (b) takes place with probability $p$. 
A discrete manifold of dimension $d=2$ is a simplicial complex formed by triangles such that every link is incident to at most two triangles.

Therefore the emergent network geometry for our model with $m=2$ is a discrete 2d manifold.
Scale-free networks

In the case $m = \infty$
a scale-free network with high clustering, significant community structure, finite spectral dimension is generated.
Planar for $p=0$. 
Degree distribution

• For $m=2$ and $p=0$ we can calculate the degree distribution given by

$$P(k) = \frac{1}{2} \left(\frac{2}{3}\right)^{k-1}$$

• For $m = \infty$ and $p=0$ we can calculate the degree distribution given by

$$P(k) = \frac{12}{(k + 2)(k + 1)k}$$
Combinatorial Curvature

The combinatorial curvature for a node $i$ of a planar triangulation is

$$R_i = 1 - \frac{k_i}{2} + \frac{T_i}{3}$$

- $k_i$ is the degree of the node $i$,
- $T_i$ is the number of triangles to which node $i$ belongs

For a node in the bulk

$$R_i = \frac{6 - k_i}{6}$$

For a node at the surface

$$R_i = \frac{4 - k_i}{6}$$
Emergent network geometry
and curvature distribution

Emergent network geometry and curvature distribution

Exponential network

\[ \langle R \rangle = \frac{1}{N} \]
\[ \langle R^2 \rangle < \infty \]

Broad degree distribution

\[ m = 2 \quad p = 0.9 \]

\[ m = 4 \quad p = 0.9 \]

\[ m = \infty \quad p = 0 \]

Scale-free network

\[ \langle R \rangle = \frac{1}{N} \]
\[ \langle R^2 \rangle = \infty \]
Finite spectral dimension

\[ L_{ij} = k_i \delta_{ij} - a_{ij} \]

\[ \rho(\lambda) \approx \lambda^{(d/2 - 1)} \]

\[ P_c(\lambda) \approx \lambda^{d/2} \]
Properties of emergent network geometries

- Small world
- Finite clustering
- High modularity
- Finite spectral dimension

Which are properties of many real network datasets.
## Properties of real datasets

<table>
<thead>
<tr>
<th>Datasets</th>
<th>$N$</th>
<th>$L$</th>
<th>$\langle \ell \rangle$</th>
<th>$C$</th>
<th>$M$</th>
<th>$d_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1L8W (protein)</td>
<td>294</td>
<td>1608</td>
<td>5.09</td>
<td>0.52</td>
<td>0.643</td>
<td>1.95</td>
</tr>
<tr>
<td>1PHP (protein)</td>
<td>219</td>
<td>1095</td>
<td>4.31</td>
<td>0.54</td>
<td>0.638</td>
<td>2.02</td>
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<tr>
<td>1AOP chain A (protein)</td>
<td>265</td>
<td>1363</td>
<td>4.31</td>
<td>0.53</td>
<td>0.644</td>
<td>2.01</td>
</tr>
<tr>
<td>1AOP chain B (protein)</td>
<td>390</td>
<td>2100</td>
<td>4.94</td>
<td>0.54</td>
<td>0.685</td>
<td>2.03</td>
</tr>
<tr>
<td>Brain-(coactivation) $^{45}$</td>
<td>638</td>
<td>18625</td>
<td>2.21</td>
<td>0.384</td>
<td>0.426</td>
<td>4.25</td>
</tr>
<tr>
<td>Internet $^{46}$</td>
<td>22963</td>
<td>48436</td>
<td>3.8</td>
<td>0.35</td>
<td>0.652</td>
<td>5.083</td>
</tr>
<tr>
<td>Power-grid $^{38}$</td>
<td>4941</td>
<td>6594</td>
<td>19</td>
<td>0.11</td>
<td>0.933</td>
<td>2.01</td>
</tr>
<tr>
<td>Add Health (school) $^{61}$ $^{47}$</td>
<td>1743</td>
<td>4419</td>
<td>6</td>
<td>0.22</td>
<td>0.741</td>
<td>2.97</td>
</tr>
</tbody>
</table>
The generalized degree $k_{d,\delta}(\mu)$ of a $\delta$-face $\mu$ in a $d$-dimensional simplicial complex is given by the number of $d$-dimensional simplices incident to the $\delta$-face $\mu$. 

$\begin{align*}
\text{Number of triangles incident to the node } & \mu \\
& k_{2,0}(\mu) \\
\text{Number of triangle incident to the link } & \mu \\
& k_{2,1}(\mu)
\end{align*}$
Generalized degree

The generalized degree $k_{d,\delta}(\mu)$ of a $\delta$-face $\mu$ in a $d$-dimensional simplicial complex is given by the number of $d$-dimensional simplices incident to the $\delta$-face $\mu$. 

<table>
<thead>
<tr>
<th>i</th>
<th>$k_{2,0}(i)$</th>
<th>(i,j)</th>
<th>$k_{2,1}(i,j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>(1,2)</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>(1,3)</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>(1,4)</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>(1,5)</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>(2,3)</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>(3,4)</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3,5)</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3,6)</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(5,6)</td>
<td>1</td>
</tr>
</tbody>
</table>
Incidence number

To each \((d-1)\)-face \(\mu\) we associate the incidence number

\[ n_\mu = k_{d,d-1}(\mu) - 1 \]

<table>
<thead>
<tr>
<th>((i,j))</th>
<th>(n_{(i,j)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,2)</td>
<td>0</td>
</tr>
<tr>
<td>(1,3)</td>
<td>2</td>
</tr>
<tr>
<td>(1,4)</td>
<td>0</td>
</tr>
<tr>
<td>(1,5)</td>
<td>0</td>
</tr>
<tr>
<td>(2,3)</td>
<td>0</td>
</tr>
<tr>
<td>(3,4)</td>
<td>0</td>
</tr>
<tr>
<td>(3,5)</td>
<td>1</td>
</tr>
<tr>
<td>(3,6)</td>
<td>0</td>
</tr>
<tr>
<td>(5,6)</td>
<td>0</td>
</tr>
</tbody>
</table>
If \( n_\mu \) takes only values \( n_\mu = 0, 1 \) each \((d-1)\)-face is incident at most to two \( d \)-dimensional simplices. 

*In this case the simplicial complex is a discrete manifold.*
Network Geometry with Flavor

Starting from a single d-dimensional simplex

(1) **GROWTH**: 
At every timestep we add a new node d simplex (formed by one new node and an existing (d-1)-face).

(2) **ATTACHMENT**: 
The probability that a new node will be connected to a face α depends on the flavor s=-1,0,1 and is given by

$$\prod_{\mu}^{[s]} = \frac{1 + s n_{\mu}}{\sum_{\mu'} (1 + s n_{\mu'})}$$

Bianconi & Rahmede (2016)
Attachment probability

\[
\Pi^{[s]}_{\mu} = \frac{(1 + s n_{\mu})}{\sum_{\mu' \in \mathcal{Q}_{d,d-1}} (1 + s n_{\mu'})} = \begin{cases} 
\frac{(1 - n_{\mu})}{Z^{[-1]}}, & s = -1 \\
\frac{1}{Z^{[0]}}, & s = 0 \\
\frac{k_{d,d-1}(\mu)}{Z^{[1]}}, & s = 1
\end{cases}
\]

- \( s = -1 \)  Manifold
- \( s = 0 \)  Uniform attachment
- \( s = 1 \)  Preferential attachment

\( n_{\mu} = 0,1 \)

\( n_{\mu} = 0,1,2,3,4... \)
Dimension $d=1$

- **Manifold**
- **Uniform attachment**
- **Preferential attachment**

- **Chain**
- **Exponential**
- **Scale-free BA model**
Dimension $d=2$

Manifold

Uniform attachment

Preferential attachment

Exponential

Scale-free

Scale-free
Dimension $d=3$

Manifold  Uniform attachment  Preferential attachment

Scale-free  Scale-free  Scale-free
Effective preferential attachment in $d=3$

Node $i$ has generalized degree 3
Node $i$ is incident to 5 unsaturated faces

Node $i$ has generalized degree 4
Node $i$ is incident to 6 unsaturated faces
Degree distribution

For \( d+s=1 \)

\[
P_d(k) = \left( \frac{d}{d+1} \right)^{k-d} \frac{1}{d+1}
\]

For \( d+s>1 \)

\[
P_d(k) = \frac{d + s}{2d + s} \frac{\Gamma(1+(2s+s)(d+s-1))}{\Gamma(d/(d+s-1))} \frac{\Gamma(k-d+d/(d+s-1))}{\Gamma(k-d+(2d+s)(d+s-1))}
\]

NGF are always scale-free for \( d>1-s \)

- For \( s=1 \) NGF are always scale free
- For \( s=0 \) and \( d>1 \) the NGF are scale-free
- For \( s=-1 \) and \( d>2 \) the NGF are scale-free
Degree distribution of NGF
Modularity and Clustering coefficient of NGF

<table>
<thead>
<tr>
<th>M</th>
<th>s = − 1</th>
<th>s = 0</th>
<th>s = 1</th>
<th>C</th>
<th>s = − 1</th>
<th>s = 0</th>
<th>s = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>d = 2</td>
<td>0.97</td>
<td>0.94</td>
<td>0.90</td>
<td>d = 2</td>
<td>0.65</td>
<td>0.74</td>
<td>0.79</td>
</tr>
<tr>
<td>d = 3</td>
<td>0.91</td>
<td>0.85</td>
<td>0.80</td>
<td>d = 3</td>
<td>0.77</td>
<td>0.81</td>
<td>0.84</td>
</tr>
</tbody>
</table>
Master equation approach

A master equation can be written for the number of $\delta$ faces $N_{d,\delta}^t(k)$ that have generalized degree $k$ at time $t$

$$
N_{d,\delta}^{t+1}(k) = \begin{cases} 
N_{d,\delta}^t(k) + m_{d,\delta}(k-1)N_{d,\gamma}^t(k-1) - m_{d,\delta}(k)N_{d,\delta}^t(k) & \text{for } k \neq m \\
N_{d,\delta}^t(k) - m_{d,\delta}(k)N_{d,\delta}^r(k) + 1 & \text{for } k = m
\end{cases}
$$

with

$$
m_{d,\delta}(k) = \frac{(1-s) + (d + s - \delta - 1)k}{(d + s)t}
$$

indicating the probability that a $\delta$ face increases its generalized degree at time $t$
Generalized degree distributions

<table>
<thead>
<tr>
<th>flavor</th>
<th>$s = -1$</th>
<th>$s = 0$</th>
<th>$s = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta = d - 1$</td>
<td>Bimodal</td>
<td>Exponential</td>
<td>Power-law</td>
</tr>
<tr>
<td>$\delta = d - 2$</td>
<td>Exponential</td>
<td>Power-law</td>
<td>Power-law</td>
</tr>
<tr>
<td>$\delta \leq d - 3$</td>
<td>Power-law</td>
<td>Power-law</td>
<td>Power-law</td>
</tr>
</tbody>
</table>

The power-law generalized degree distribution are scale-free for

$$d \geq d_c^{[\delta,s]} = 2(\delta + 1) + s$$
Emergent Hyperbolic geometry

The emergent hidden geometry is the hyperbolic $H^d$ space. Here all the links have equal length.
Emergent hyperbolic geometry

\[ d = 3 \]
The pseudo-fractal geometry of the surface of the 3d manifold (random Apollonian network) is connected with the Apollonian network.
NGF and Apollonian networks

Network Geometry with Flavor (NGF)
\[ s = -1 \quad d = 3 \]

Apollonian random network
Planar projection of the NGF
Spectral dimensions of NGF with $s=-1$

\[ L_{ij} = k_i \delta_{ij} - a_{ij} \]
\[ \rho(\lambda) \approx \lambda^{-(d_s/2-1)} \]
\[ P_c(\lambda) \approx \lambda^{-d_s/2} \]
Not all the nodes are the same!

Let assign to each node $i$ an energy $\varepsilon$ from a $g(\varepsilon)$ distribution.
Energy of the $\delta$-faces

Every $\delta$-face $\alpha$ is associated to an energy which is the sum of the energy of the nodes belonging to $\alpha$

$$\mathcal{E}_\alpha = \sum_{i \subseteq \alpha} \mathcal{E}_i$$

For example, in $d=3$

the energy of a link $\varepsilon_1$ \hspace{1cm} $\varepsilon_2$ is $\varepsilon_1 + \varepsilon_2$

the energy of a face $\varepsilon_1$ \hspace{1cm} $\varepsilon_2$ \hspace{1cm} $\varepsilon_3$ is $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$
Starting from a single $d$-dimensional simplex

**(1) GROWTH**

At every timestep we add a new node $d$ simplex
(form by one new node and an existing $(d-1)$-face).
The new node has energy $\varepsilon$ drawn from the distribution $g(\varepsilon)$

**(2) ATTACHMENT**

The probability that a new node will be connected to a face $\alpha$ depends on the flavor $s=-1,0,1$ and is given by

$$\prod_{\mu}^{[s]} = \frac{e^{-\beta \varepsilon \mu} (1 + s n_{\mu})}{\sum_{\mu'} e^{-\beta \varepsilon \mu} (1 + s n_{\mu'})}$$

Bianconi & Rahmede (2016)
NGF
with flavor s=-1
Manifolds
Manifolds in $d=3$

In NGF with $s=-1$ and $d=3$
also called
Complex Quantum Network Manifolds
the average of the generalized degree follow
the Fermi-Dirac, Boltzmann and Bose-Einstein
distribution
respectively for
triangular faces, links and nodes
The average of the generalized degree of the NGF over $\delta$-faces of energy $\epsilon$

$$\left\langle \left[ k_{d,\delta} - 1 \right] \right\rangle \epsilon$$

follows a regular pattern

<table>
<thead>
<tr>
<th>$\delta = d - 1$</th>
<th>Fermi-Dirac</th>
<th>Boltzmann</th>
<th>Bose-Einstein</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta = d - 2$</td>
<td>Boltzmann</td>
<td>Bose-Einstein</td>
<td>Bose-Einstein</td>
</tr>
<tr>
<td>$\delta \leq d - 3$</td>
<td>Bose-Einstein</td>
<td>Bose-Einstein</td>
<td>Bose-Einstein</td>
</tr>
</tbody>
</table>
Emergent geometry at high temperature

$d=2$

$\beta=0.01$
Emergent geometry at low temperature

d=2
β=5
Emergent geometry at high temperature

$d=3$
$\beta=0.01$
Emergent geometry at low temperature

d=3
β=5
Conclusions

**Emergent complex network geometries**

- Show small world behavior, finite clustering coefficient, high modularity, finite spectral dimension which are properties of many real network datasets.
- Display a distribution of the local curvature that can be exponential or scale-free.
- For m=2 they generate random manifolds.

**Network Geometry with Flavor**

- Are scale-free for d>1 also when they do not include an explicit preferential attachment. The dimension d=3 is the lowest dimension for obtaining scale-free manifold.
- They have generalized degrees displaying Fermi-Dirac, Boltzmann or Bose-Einstein distribution depending on the dimensionality of the δ-face and on the flavour s.
Collaborators and References

Emergent network geometry

G. Bianconi C. Rahmede Network geometry with flavor PRE 93, 032315 (2016)
G. Bianconi and C. Rahmede Scientific Reports 7, 41974 (2017)
O. T. Courtney and G. Bianconi PRE 95, 062301(2017)

Ensembles of simplicial complexes

O. T. Courtney and G. Bianconi PRE 93, 062311 (2016)
A primer on Network Theory
Complex Networks

describe

the interactions between the elements of large complex

Biological, Social and Technological systems.
Complexity: between randomness and order

LATTICES

Simple cubic
Simple tetragonal
Simple orthorhombic
Rhombohedral

Face-centered cubic
Body-centered tetragonal
Body-centered orthorhombic
Base-centered orthorhombic
Triclinic

COMPLEX NETWORKS

Scale free networks
Small world
With communities
ENCODING INFORMATION IN THEIR STRUCTURE

RANDOM GRAPHS

Totally random
Binomial degree distribution

Regular networks
Symmetric
Scale-free networks

\[ P(k) \propto k^{-\gamma} \quad \gamma \in (2,3] \]

\[ \langle k \rangle \text{ finite} \]

\[ \langle k^2 \rangle \rightarrow \infty \]
Scale-free networks

- Technological networks
  - Internet
  - World-Wide Web
- Biological networks
  - Metabolic networks
  - Protein-interaction networks
  - Transcription networks
- Transportation networks
  - Airport networks
- Social networks
  - Collaboration networks
  - Citation networks
  - Facebook
- Economical networks
  - Networks of shareholders
  - The World Trade Web
**BA scale-free model**

(1) **GROWTH**
At every timestep we add a new node with \( m \) edges (connected to the nodes already present in the system).

(2) **PREFERENTIAL ATTACHMENT**
The probability \( \Pi(k_i) \) that a new node will be connected to node \( i \) depends on the connectivity \( k_i \) of that node:

\[
\Pi(k_i) = \frac{k_i}{\sum_j k_j}
\]

Growth by uniform attachment of links

(1) **GROWTH**:
At every timestep we add a new node with $m$ edges (connected to the nodes already present in the system).

(2) **UNIFORM ATTACHMENT**:
The probability $\Pi_i$ that a new node will be connected to node $i$ is *uniform*

$$\Pi_i = \frac{1}{N}$$

Growing networks
and quantum statistics
The unitary cell of growing networks

Consider the following “unitary cell” of a network

To grow a network need to attach the unitary cell to the exiting network. We have two options
The unitary cell of growing networks

Consider the following “unitary cell” of a network

To grow a network need to attach the unitary cell to the exiting network. We have two options

An old node is attached to m new nodes
The unitary cell of growing networks

Consider the following “unitary cell” of a network

To grow a network need to attach the unitary cell to the exiting network. We have two options

- An old node is attached to $m$ new nodes
- A new node is attached to $m$ old nodes
Not all the nodes are the same!

Let assign to each node an energy $\varepsilon$ from a $g(\varepsilon)$ distribution that describes an intrinsic quality of the node
Fitness

The fitness of a node $i$ is given by

$$\eta_i = e^{-\beta \varepsilon_i}$$

where $\beta=1/T$ is the inverse temperature.

If $\beta=0$ all the nodes have same fitness.
If $\beta \gg 1$ small differences in energy have large impact on the fitness of the faces.
The Complex Growing Cayley tree model

**Growth:**

– At each time attach a old node with $\rho_i=1$ to $m$ links are added to the network and then we set $\rho_i=0$.
– To each node $i$ we assign a energy $\varepsilon_i$ from a $g(\varepsilon)$ distribution

**Attachment towards high energy nodes:**

– The node $i$ to which we attach the new “unitary cell” is chosen with probability

$$\Pi_i = \frac{e^{-\beta \varepsilon_i} \rho_i}{\sum_{j} e^{-\beta \varepsilon_j} \rho_j}$$
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$$
MF Equations for the growing scale-free network and the complex growing Cayley tree network

• Scale-free Bianconi-Barabasi model

• Complex Growing Cayley tree
Classical and quantum statistics

**Classical particles**
Boltzmann statistics-\( (s=0) \)
occupation number \( n=0,1,2,\ldots \)

**Quantum particles**
- Fermi particles-\( (s=-1) \)
ockupation number \( n=0,1 \)
- Bose particles-\( (s=1) \)
ockupation number \( n=0,1,2\ldots \)

Average occupation number

\[
n(\varepsilon) = \frac{1}{e^{\beta(\varepsilon - \mu)} + S}
\]
Solution of the Bianconi-Barabasi model

The average degree of node increases in time as a power-law with an exponent depending on its energy, $\beta$ and a self-consistent constant $\mu_B$

$$\bar{k}_i(t) = m \left( \frac{t}{t_i} \right)^{f_B(\epsilon)}$$

$$f_B(\epsilon) = e^{-\beta(\epsilon - \mu_B)}$$

The self consistent constant $\mu_B$ is determined by the same equation fixing the chemical potential in a Bose gas!

$$1 = \int d\epsilon \ g(\epsilon) \frac{1}{e^{\beta(\epsilon - \mu_B)} - 1}$$
Solution of the complex growing Cayley model

The average $\rho$ of node (determining the probability that a node is at the interface) decreases in time as a power-law with an exponent depending on its energy, $\beta$ and a self-consistent constant $\mu_F$

$$\bar{\rho}_i(t) = m\left(\frac{t}{t_i}\right)^{-f_F(\varepsilon)}$$

$$f_F(\varepsilon) = e^{-\beta(\varepsilon - \mu_F)}$$

The self consistent constant $\mu_F$ is determined by the same equation fixing the chemical potential in a Fermi gas!

$$\frac{1}{m} = \int d\varepsilon \ g(\varepsilon) \frac{1}{e^{\beta(\varepsilon - \mu_F)} + 1}$$
The Bianconi-Barabasi model

Growth:
– At each time a new node and $m$ links are added to the network.
– To each node $i$ we assign an energy $\varepsilon_i$ from a $g(\varepsilon)$ distribution

Preferential attachment towards high degree low energy nodes:
– Each node connects to the rest of the network by $m$ links attached preferentially to well connected, low energy nodes.

\[
\Pi_i = \frac{e^{-\beta \varepsilon_i k_i}}{\sum_j e^{-\beta \varepsilon_j k_j}}
\]
Bose-Einstein condensation in complex networks

Scale-Free Fit-get-rich Phase

\[ \beta < \beta_C \]

Bose-Einstein condensate Phase

\[ \beta > \beta_C \]

G. Bianconi, A.-L. Barabási 2001
Energy distribution of the nodes at the bulk of the growing Cayley tree network

G. Bianconi (2002)