FINITE ELEMENT STUDIES OF REINFORCED AND
UNREINFORCED TWO-LAYER SOIL SYSTEMS

by

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ABSTRACT

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The purpose of this study is to obtain an insight into the mechanisms by which a geosynthetic membrane influences the performance of a plane strain and an axisymmetric two-layer soil system, where the reinforcement is incorporated either into a layer of fill, or at the interface of a layer of fill overlying clay subgrade.

New axisymmetric membrane and interface element formulations are developed and incorporated into an existing large strain finite element code. A linear elastic model of behaviour is used for the membrane material and an elastic-perfectly frictional model, based on the Mohr-Coulomb yield function, is implemented for the interface. These new formulations both take account of large global displacement and rotation effects, although the interface element is constrained to small relative displacements, and are checked against small and large strain closed form test problems. The finite element equations are based on an Updated Lagrangian description of deformation.

Plane strain finite element investigations into the significance of the resolution and relative size of the finite element mesh, and the differences between large and small strain analyses, are undertaken. For typical unreinforced and reinforced plane strain and axisymmetric two-layer soil systems a detailed analysis is presented of the soil dislocations, strains, stresses, principal stress directions, mobilised fill friction angles and the stresses on the underside of the footing. A series of plane strain and some axisymmetric parametric studies of the various material properties is conducted, to assess the influences and relative importance of those variables to the performance of the two-layer soil system under monotonic loading. The study considers various reinforcement lengths and stiffnesses, fill depths and strengths, and different clay strengths.

The mechanisms of reinforcement are identified through careful examination of the footing load-displacement response, the reinforcement tension and the stresses and displacements at the interfaces with the surrounding soil. A comparative study is also undertaken between the results obtained by the finite element model and those predicted by a plane strain and axisymmetric limit equilibrium design method.

The effects of including a low friction membrane within an oil storage tank base, as secondary containment against oil leakage, are investigated by a series of axisymmetric finite element analyses.
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Only for you, children of doctrine and learning, have we written this work. Examine this book, ponder the meaning we have dispersed in various places and gathered again; what we have concealed in one place we have disclosed in another, that it may be understood by your wisdom.

Heinrich Cornelius Agrippa von Nettesheim, *De occulta philosophia*, 3, 65

To my parents
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References
Nomenclature:

Additional symbols and occasional alternative usage of the symbols below are defined where they occur in the text.

\( A \)  Cross-sectional area of membrane element
\( a_i \)  Generalised reference co-ordinate
\( [B] \)  Matrix relating strain rate and nodal velocities
\( B \)  Footing half width in plane strain conditions
\( B' \)  Effective footing half width
\( [C] \)  Distortion rate matrix
\( [D] \)  Material stiffness matrix
\( [D]^e \)  Elastic material stiffness matrix
\( [D]^p \)  Plastic material stiffness matrix
\( [D]^{ep} \)  Elasto-plastic material stiffness matrix
\( D_c \)  Clay depth
\( D_f \)  Fill depth
\( E, E_u \)  Young's modulus
\( F_r \)  Longitudinal reinforcement tension
\( F_{tang} \)  Tangential reinforcement tension
\( F_{circ} \)  Circumferential reinforcement tension
\( F_u \)  Ultimate bearing load
\( F_{ix}, F_{iy} \)  Nodal forces
\( f_i \)  Nodal shape function
\( f(\sigma) \)  Yield function
\( G \)  Shear modulus
\( G_c \)  Clay shear modulus
\( G_f \)  Fill shear modulus
\( g(\sigma) \)  Plastic potential
\( H_p \)  Depth of plastic failure zone in clay
\( [J] \)  Jacobian matrix
\( J \)  Reinforcement stiffness
\( [K] \)  Stiffness matrix
\( K \)  Reinforcement bulk modulus
Nomenclature:

\( \mathbf{K} \) Coefficient of punching shearing resistance
\( K_0 \) Coefficient earth pressure
\( k \) Vector constant
\( k_s \) Interface element shear stiffness
\( k_n \) Interface element normal stiffness
\( L \) Reinforcement length
\( L^m \) Length of membrane element
\( L_e \) Characteristic length of interface element
\( [N] \) Shape function matrix
\( N_c \) Bearing capacity factor due to cohesion
\( N_i \) Bearing capacity factor due to self-weight
\( P \) Average footing pressure
\( P_f \) Final footing pressure
\( P_i \) Generalised nodal force
\( P_p \) Total passive earth pressure
\( P_u \) Ultimate bearing capacity
\( P \) Vector of nodal forces
\( q \) Surcharge
\( R \) Footing radius in axisymmetric conditions
\( R_i \) Unbalanced nodal load
\( R \) Vector of global nodal co-ordinates
\( r \) Radius
\( r' \) Mapped co-ordinate vector, in terms of \( \alpha \)
\( s_u \) Undrained shear strength of clay
\( s_u'^{tr} \) Undrained triaxial shear strength of clay
\( s_u^{ps} \) Undrained plane strain shear strength of clay
\( T \) Instantaneous membrane element thickness
\( t \) Time
\( t_0 \) Initial reinforcement thickness
\( U \) Vector of nodal velocities
\( \ddot{U}_i \) Generalised nodal velocity
\( u' \) Mapped velocity vector, in terms of \( \alpha \)
\( u_r \) Relative tangential velocity/slip
\( v_r \) Relative normal velocity/separation
\( W_c \) Clay width
\( W_p \) Width of plastic failure zone in clay
\( X \)  
Vector of global nodal co-ordinates

\( \bar{x} \)  
Horizontal distance from the footing centre-line

\( x_i \)  
Generalised co-ordinate

\( \alpha, \beta \)  
Isoparametric reference co-ordinates

\( \beta \)  
Load spread angle through the fill

\( \delta \)  
Vertical footing displacement

\( \delta_f \)  
Angle of friction on the footing base

\( \dot{\varepsilon} \)  
Vector of strain rates

\( \dot{\varepsilon}_l \)  
Longitudinal Hencky strain rate

\( \dot{\varepsilon}_t \)  
Tangential Hencky strain rate

\( \dot{\varepsilon}_\theta \)  
Circumferential Hencky strain rate

\( \phi_{ps} \)  
Plane strain friction angle of fill

\( \phi_r \)  
Triaxial friction angle of fill

\( \phi_{sw} \)  
Critical state friction angle

\( \phi_i \)  
Interface element friction angle

\( \phi' \)  
Mobilised friction angle

\( \phi'_i \)  
Mobilised friction angle at interface

\( \gamma_s \)  
Degree-of-association

\( \gamma_c \)  
Bulk unit weight of clay

\( \gamma_f \)  
Bulk unit weight of fill

\( \nu_c \)  
Poisson's ratio of clay

\( \nu_f \)  
Poisson's ratio of fill

\( \nu_r \)  
Poisson's ratio of reinforcement

\( \Theta \)  
Rotation angle

\( \sigma \)  
Normal stress

\( \sigma_L \)  
Longitudinal stress

\( \sigma_t \)  
Tangential stress

\( \sigma_\theta \)  
Circumferential stress

\( \sigma_v \)  
Vertical stress

\( \sigma \)  
Vector of stresses

\( \sigma^i \)  
Imposed normal footing stress on reinforcement

\( \tau \)  
Shear stress

\( \tau_f \)  
Shear stress at the footing base

\( \psi_{ps} \)  
Plane strain dilation angle of the fill

\( \psi_i \)  
Interface element dilation angle
CHAPTER 1:

INTRODUCTION

1.1 Geotextiles and Geomembranes

Geotextiles and geomembranes, broadly speaking, are synthetic fibres used to stabilise structures built of earth. The now widely accepted generic term for these non-natural materials is ‘geosynthetics’, which was coined by Dr. J. P. Giroud (1986) in his extensive and exhaustive review of geotextiles. Geosynthetics include permeable and impermeable materials that are either of a knitted, woven, or non-woven nature, as well as polymer grids and meshes.

There are many different practical applications of geosynthetics in civil engineering. They are often incorporated into geotechnical constructions, such as steep slopes, retaining walls, embankments and roads, in order to provide strength and reinforcement, or they can be used for erosion control, rockfall nets and gabion cages to protect the construction from weathering and deterioration. They are used also to control fluid transmission, either as impermeable liners to prevent leakage from beneath fluid storage tanks, canals, reservoirs, dams and waste disposal sites, or as permeable filters allowing fluid to pass through the system, but still retaining the soil on the upstream side.

The role of the geosynthetic membrane varies in each of the above applications. It can operate in the functions of Reinforcement, Separation, Filtration, Protection, Containment and Fluid Transmission. A comprehensive discussion and evaluation of these functions is given by both Koerner (1986) and John (1987).
Clearly, geosynthetic materials have become an integral part of geotechnical engineering constructions and the specific application considered in this dissertation is that of two-layer soil systems. The use of geosynthetics within two-layer soil systems is well-established, utilising the functions of reinforcement, separation, containment and filtration. Their construction typically consists of a geosynthetic material laid on the stripped soft clay subgrade with a layer of aggregate fill compacted on top, and their uses are multitudinous, for example, temporary working platforms, car parks, outlying aircraft runways, foundations of storage tanks and silos, forestry trails, haul roads, access ways and low-maintenance unpaved roadways. The majority of previous research into two-layer soil systems, by experimental and theoretical methods, has addressed the specific problem of reinforced unpaved road design, Figure 1.1, as reviewed in Chapter 2.

![Figure 1.1: A Typical Reinforced Unpaved Road Section](image)

### 1.2 Objectives of Research

The work described in Chapters 2 to 10 of this thesis has four main objectives. Firstly, to develop further the existing large strain finite element code OXFEM (initially developed by Burd (1986)) to include both plane strain and axisymmetric frictional interface elements, as well as axisymmetric membrane elements, so that the soil-reinforcement interaction of a two-layer soil system can be modelled accurately in both plane strain and axisymmetry. Secondly, to investigate and analyse numerically the mechanisms by which a geosynthetic membrane reinforces a plane strain and an axisymmetric system for large surface displacements. Thirdly, to conduct a detailed parametric study for each of the important
material properties and dimensions of a two-layer soil system and assess their influence on its performance, and finally, to evaluate the usefulness of the Houlsby et al. (1989) and Houlsby and Jewell (1990) analyses as appropriate design methods.

It is worthy of note that the plane strain results of the parametric study pre-date the development of the new axisymmetric membrane and interface elements and therefore comprise a greater proportion of the reported parametric analyses. An example of a practical application of the new axisymmetric interface elements to the analysis of the use of membranes in granular base foundations beneath oil storage tanks is described in Chapter 11.
CHAPTER 2:

ANALYSIS AND DESIGN OF REINFORCED TWO-LAYER SOIL SYSTEMS

2.1 Introduction

This chapter considers the problem of the design and analysis of reinforced two-layer soil systems, with particular reference to their common use as reinforced unpaved roads (Figure 1.1).

A synopsis of the different reinforcement mechanisms that have been identified as operating in a reinforced two-layer soil system is given in Section 2.2. This is followed by two reviews, firstly of the previous research conducted on model and full-scale two-layer soil systems, Section 2.3, and secondly of several finite element models specifically designed to analyse and investigate such reinforced soil systems, given in Section 2.4. Finally, Section 2.5 gives an overview of the currently available simple design methods for reinforced two-layer soil systems and their inherent understanding of the reinforcement mechanisms involved.

A key problem in designing and analysing a reinforced unpaved road is accounting for the effect of repeated loading. The majority of the currently available design methods initially design for the case of monotonic loading and then make empirical adjustments to the calculated allowable load to account for the repeated loading case, a procedure discussed further in Section 2.5.2. This methodology is clearly limited, since there is no rigorous analysis of the mechanics involved under repetitive loads, but it is generally accepted at present that such methods are valid and they are considered, in this thesis, to be the best.
currently available. Therefore, based on the same assumptions, the research herein is concerned with understanding the mechanisms operating in a two-layer soil system subjected only to static monotonic loading.

2.2 Reinforcement Mechanisms

As a result of previous work in this area several mechanisms of reinforcement have been identified, or theorised, as acting in a reinforced two-layer soil system. These are described briefly below:

Membrane Mechanism: Tensile forces are set up in the reinforcement by the shear stresses acting at the interface with the soil. During heavy, or repeated loading the aggregate layer deforms substantially, creating surface ruts which force the geotextile to deform, see Figure 2.1. If the tensile forces are coincident with an appreciable curvature of the reinforcement, then the normal stresses in the soil acting on each side of the reinforcement are unequal. This effect tends to reduce the normal stresses transmitted to the subgrade immediately underneath the load which increases the capacity of the road. This phenomenon is known as the tensioned membrane effect.

![Diagram of Reinforcement Mechanisms](image)

**Figure 2.1: Assumed Profile of the Deformed Geotextile**

*(After Giroud and Noiray (1981))*
Restraint Mechanisms: Lateral movements at the base of the aggregate layer and the surface of the subgrade are restrained by the reinforcement, which increases the strength and stiffness of the structure. This is achieved in two ways:

1) Through friction and/or interlock between the reinforcement and the surrounding soil, tensile strains at the base of the fill will be reduced. This mechanism effectively strengthens the fill layer by restricting any slip at the fill-reinforcement interface.

2) When the reinforcement deformation becomes appreciable it tends to restrain the heave movements of the subgrade on each side of the load through the membrane mechanism associated with the reversed curvature of the reinforcement. This effectively contributes an additional surcharge loading to the subgrade surface which will enhance its loading capabilities underneath the load.

Shear Stress Mechanism: As a vertical load is applied to the two-layer soil system lateral stresses develop in the fill layer which generate shear stresses at the base of the fill, see Figure 2.2. If reinforcement is absent these shear stresses are sustained directly by the subgrade and have the detrimental effect of lowering its bearing capacity, (Bolton (1979)). In the reinforced case the shear stresses at the fill base are assumed to be sustained by the reinforcement, rather than the subgrade, which causes a corresponding increase in the bearing capacity of the subgrade for vertical loads.

Improved Load Spread: By including a geotextile reinforcement at the base of the fill layer it is a commonly held belief that the wheel load distributes itself over a wider area on the subgrade surface, compared to the area under pressure in the unreinforced case. This wider load distribution would improve the subgrade bearing capacity. However, it is noteworthy that from the results presented herein this mechanism is not generally present.

Separation: A geotextile will prevent the loss of subbase material into the subgrade on a large scale, thereby ensuring retention of the full design thickness of the subbase even at large deformations. Through this continuity the subbase layer will fully maintain its mechanical properties. Geotextile reinforcement may act as a valuable separating layer on very soft subgrade and particularly under cyclic loading, since it is in these situations that
progressive deterioration of the two-layer soil system will be increased due to the larger magnitude and/or rate of action of such detrimental mechanisms as the migration and mixing of the fill and soil particles and lateral movement of the subbase layer.

2.3 Previous Model and Full-Scale Studies

Numerous experimental studies on the problem of the design of reinforced two-layer soil systems, specifically reinforced unpaved roads, have been carried out in recent years. Several of these studies, Love (1984), Poran (1985), Fannin (1986) and Little (1992), are discussed in more detail below.

High quality model tests were conducted by Love (1984) to investigate the performance of a model geogrid in a quarter full-scale study of a reinforced unpaved road. These test results are summarised by Milligan and Love (1984), while Love et al. (1987) compares and discusses the experimental model results with the corresponding finite element modelling conducted by Burd (1986), which is described in Section 2.4. Simple plane strain monotonically loaded footing tests were carried out on systems consisting of a fill layer compacted onto a consolidated clay subgrade, both with and without a model grid placed
at their interface. Eighteen main tests were carried out for combinations of three different fill thicknesses \( D_f = 1.333B, 2.9B \) and \( 2.667B \) where \( B \) is the footing half width = 37.5 mm placed over clay of three different nominal strengths (6 kPa, 9 kPa and 14 kPa).

The resulting load-penetration data given by Love (1984) shows that the limit load, defined at a footing penetration of 50 mm, in each of the unreinforced cases is dependent on the clay strength, but virtually independent of the fill thickness, whereas in the reinforced cases the limit load is dependent on both clay strength and fill depth. It was further noted that in the initial elastic stages of testing there was very little difference between the reinforced and unreinforced systems, whereas in the plastic regimes of the load-penetration curves the reinforced responses were significantly stiffer compared to the unreinforced responses, an effect seen in all of the finite element parametric pressure-displacement responses presented later in this thesis. Love (1984) also showed that the zone of deformation in a reinforced test could be seen generally to extend approximately twice as deep into the clay as for an unreinforced test with the same fill thickness, clay strength and footing penetration (but a much smaller load). Additionally there was found to be much less fill material lost from immediately beneath the footing in the reinforced cases compared to the unreinforced cases where lateral movement of fill material occurred along the unreinforced fill-clay interface.

The explanation of why an increase in fill thickness does not significantly change the amount of load the unreinforced road can carry, is that the generated shear stresses at the fill base are also increased and they directly reduce the clay's load bearing capacity (the 'shear stress mechanism' in Section 2.2). Consequently the unreinforced road can only maintain an approximately constant ultimate load, despite the larger fill thickness, for a particular clay strength. In contrast, with the inclusion of a grid reinforcement there is a marked increase in the load carrying capacity of the system, which improves with increases of either the clay strength, or the fill depth. This is because the membrane prevents the detrimental shear stresses acting directly upon the clay surface and thus allows the clay to provide the full bearing capacity.
Large-scale plate loading laboratory investigations were carried out by Poran (1985) on soil-geogrid interaction, related to the bearing capacity of reinforced granular layers overlaying soft clay. The failure load, defined at the point where the vertical displacement of the square plate continued to increase throughout the first ten minutes after the load application, was attained in three loading/unloading cycles for each test. Poran (1985) found that Tensar grid reinforcement improves the load bearing capacity of such structures by 40-100% and reduces the vertical settlement, in comparison to the unreinforced, under relatively heavy loads. Different failure modes were observed for the unreinforced and reinforced tests. In the unreinforced tests significant punching of the plate into the base layer occurred, while no punching developed in the reinforced tests. This effect is mostly attributed by Poran (1985) to the confinement of the base material by the reinforcement (i.e. the restraint mechanism 1 discussed in Section 2.2), with an additional mechanism acting of an improved and more uniform load spread through the aggregate. The tensioned membrane effect is considered only important for very large displacements and therefore insignificant for normal bearing capacity applications.

Full-scale reinforced and unreinforced two-layer soil system trials, with monotonic loading of a single footing, and quarter scale model tests, for both monotonic and repeated loading using a dual footing, were performed by Fannin (1986) and discussed by Milligan et al. (1986) and (1989). The model tests showed that the failure mechanisms in the five-cycle repeated loading tests were the same as in the monotonic tests, which supports the argument for the common design practice of linking the two types of loading. In the cyclic tests the behaviour was identified as essentially one of low-cycle plastic fatigue, with deformations gradually accumulating with each cycle of loading at an ever accelerating rate until failure was reached. These repeated loading results compare well with the empirical fatigue curves derived from the extensive full-scale testing by Hammit (1970), for unreinforced unpaved roads, and by De Groot (1986), for reinforced unpaved roads, which are studies discussed further in Section 2.5.

From the full-scale monotonic tests conducted by Fannin (1986), with rectangular and circular footing plates involving an arrangement of compacted granular layers over a prepared London clay with a high strength polymer grid, it was discovered that directly
under the footing the grid maintained an interlocking action with the granular material throughout testing by an interpenetration of the particles into the grid apertures. Outside this region compaction of granular material during the preparation for testing had also caused the grid to interlock with the base of the granular layer. However, as a test proceeded upward heave movements of the deforming subgrade caused this interlock to be lost, with the grid being drawn increasingly into the clay. Fannin (1986) concluded that at moderate displacements the subgrade beyond the footing area would slide along the base of the granular layer and grid, but at large displacements it would ‘flow’ through the grid apertures.

The University of Nottingham completed a full-scale test of a geosynthetic reinforced haul road on the British Science and Engineering Research Council owned Soft Clay Site, at Bothkennar in Scotland (Little (1992)). Different geosynthetic manufacturers and the Universities of Nottingham and Oxford each designed two sections of the haul road circuit, one lower bound section with a life expectancy of 1,000 passes of an 80 kN axle load and one upper bound section for 10,000 passes, as explained in a technical note by Dawson and Little (1990). Failure by trafficking was deemed to occur at a rut depth of 150 mm, or if the geosynthetic ruptured. The Oxford sections of the haul road were designed by using the Houlsby et al. (1989) analytical method (reviewed in Section 2.5) and by running a series of plane strain, large displacement finite element analyses using OXFEM (see Section 2.4).

The final summary report by Dawson and Brown (1992) confirms that the basic mechanisms proposed by Houlsby et al. (1989), such as the shear stress mechanism, are correct, but that the tensioned membrane effect is unimportant at normal rut depths, which is in disagreement to the parametric study results presented in this thesis, where both mechanisms are found to be significant. Another important conclusion from the Bothkennar project is that internal aggregate compaction and/or the development of permanent shear strains can be a significant reason for rutting, which are factors that are almost impossible to accurately account for in an analytical or finite element study.
2.4 Finite Element Analyses

Finite element modelling has the advantages over experimental modelling that parameters may be varied easily and details of stresses and deformations throughout the system may be studied. This is particularly valuable for investigating the mechanisms acting at the unreinforced and reinforced interface between the fill and subgrade, which is extremely difficult to do in a model test.

Finite element numerical methods may be used to contribute to the design of reinforced two-layer soil systems in two general ways. Firstly, numerical models can be developed that may be used directly in the design of reinforced soil structures. Several numerical models have been developed that are capable of this function, four of these (Zeevaert (1980), Poran (1985), Kwok (1987) and Burd (1986)) are described briefly in this Section 2.4. These computer models, however, are not generally available to practising engineers and even if such numerical models were generally available it is clear that in most cases the use of such sophisticated design tools would be inappropriate, if only because of time and cost restraints. Nevertheless, finite element computer analyses can be incorporated into the design process, as discussed by Hird and Pyrah (1990). In this paper the authors describe the numerical modelling of an existing embankment reinforced at the base with a geogrid, using the finite element program CRISP. The authors consider that their finite element analysis could have played a significant role in providing guidance to the designer of the embankment. The second approach that may be adopted in the application of numerical models to design, is to use the theoretical calculation to investigate the mechanisms of reinforcement operating within a reinforced soil structure with the purpose of developing improved simple analytical design methods. While substantial progress has been made in developing analytical and limit equilibrium calculations for designing reinforced two-layer soil systems (Section 2.5), much work remains to be done to explore the deformation behaviour that occurs within them. Realistically this can be done only by employing numerical methods. The finite element studies described in this dissertation are intended to contribute to the understanding of the reinforcement mechanisms operating in a reinforced two-layer soil system in the manner of the second of these two approaches.
Zeevaert (1980) developed an axisymmetric finite element formulation which used eight node isoparametric elements to represent both the subgrade and fill, and an axisymmetric two node, linear displacement, two dimensional element to model the reinforcing material. The interfaces between the reinforcement elements and the fill above, as well as the subgrade below, are modelled by six noded linear spring elements which can account for both slip and separation of the mating surfaces. The nonlinear behaviour of the soil is described by a uniaxial stress-strain curve and the plasticity characteristics of each material are taken into account by use of the extended von Mises yield criterion proposed by Drucker and Prager (1952), which is limited to small strains, i.e. infinitesimal distortions. This yield criterion has a smooth yield surface that may be used to approximate the Mohr-Coulomb yield surfaces, but it is known to be a poor representation of real soil behaviour. Large or small displacements (i.e. rigid body movements) can be modelled by the finite element formulation through a process of updating the mesh geometry at each load increment. A ‘no-tension’ mathematical model is incorporated for the aggregate fill.

A very interesting phenomenon reported by Zeevaert (1980), is that the large displacement analyses exhibited greater footing deformations than for the small displacement option at the same footing load, and that this difference increased for decreasing fill thicknesses, i.e. a 10% larger footing displacement developed for the large displacement reinforced and unreinforced thick aggregate layer \(D_f = 2.33 B\) analysis, compared to the equivalent small displacement analysis, and a 44% increase in displacement occurred for the thin layer \(D_f = 1.5 B\), where \(B\) is the footing radius of 76 mm. This is contrary to the expected result, since a large displacement calculation accounts for the geometric non-linearities associated with the deformed system and therefore intrinsically promotes comparatively smaller deformations at higher loads, as explained and illustrated in Chapter 6 of this thesis. Therefore the finite element model presented by Zeevaert (1980) must be viewed with some strong reservations. Zeevaert (1980) does conclude, however, that the computed elastic-plastic deformations are smaller for a reinforced two-layer soil system, compared to an unreinforced, when subjected to the same footing load and that this reinforcing effect is more pronounced for a thinner aggregate layer, \(D_f = 1.5 B\), than for a thicker layer, \(D_f = 2.33 B\), as concluded also in Chapter 7. A limitation of the Zeevaert
(1980) study is that only one value of aggregate friction angle is used, whereas Chapter 7 illustrates that the amount of improvement obtained by including reinforcement is dependent upon both the aggregate thickness and the friction angle.

The finite element model proposed by Poran (1985) involves a plane straia analysis using a linear elastic 'composite layer' model for the geogrid-soil layer, where the reinforcement elements have the same properties as the surrounding granular material, but are overlaid by special beam-membrane four node elements which can model large deformation effects. The clay and sand are each modelled by almost identical discrete finite element formulations, differing only in that the clay is able to simulate consolidation and loading rate effects as necessary. The formulation is described as semi-discrete, since it does not model all of the components of the system individually, as would a fully discrete analysis (e.g. Burd (1986)), but alternatively it is not a fully composite analysis either, which would model the system as an equivalent orthotropic homogeneous continuum.

The finite element analyses presented by Poran (1985) display an interesting characteristic where the contours of vertical stress in the aggregate layer are modified by the inclusion of reinforcement, so that the fluctuations in the stress for the unreinforced case improve to a more uniform and a slightly wider distribution for the same footing displacement. This increased load spread effect, however, is not identified in the unreinforced and reinforced normal stress distribution plots and contours presented later in this thesis (e.g. Plates 6.3 and 6.4, Appendix 6B) and is thought to be a consequence of the semi-discrete formulation used. Poran (1985) makes a comparative study of the load-displacement results for the laboratory plate bearing tests (adjusted to plain strain conditions by applying relevant shape factors) and the equivalent finite element computations. A good correlation exists for the unreinforced case, but the load-displacement responses for the reinforced cases are much less comparable, because the finite element analyses predict a much stiffer response, such that the vertical displacements are estimated at about half those of the laboratory results for the same final loads.

Based on test results from investigations by Meyerhof (1974), Hanna and Meyerhof (1980), Giroud et al. (1984) and especially Milligan and Love (1984), as well as his own
experimental data, Poran (1985) proposes a simplified bearing capacity model for the analysis and design of footings placed on Tensar-geogrid reinforced granular bases overlaying soft clay. The tensioned membrane mechanism is considered insignificant by Poran (1985) and is therefore neglected, however the model does include the reinforcement effects of:

1) Confinement of the granular base and the clay so that the clay’s plastic yield limit of \((\pi + 2) s_u\) is adopted, rather than the elastic yield limit of \(\pi s_u\) which is used in the unreinforced analysis.

2) Improved load distribution angle, determined empirically by incorporating test results from both previous studies and from the Poran (1985) investigation.

On comparison with other design methods (Section 2.5) for predicting footing pressures at failure for unreinforced two-layer soil systems, the Poran (1985) model shows good agreement. For predicting the footing pressure at failure for the reinforced case, however, the model underestimates previous test results by up to 25%.

Kwok (1987) considered the slightly different two-layer soil problem of embankments placed on reinforcement over a soft foundation layer, rather than the comparatively smaller monotonically loaded two-layer soil systems as considered by Zeevaert (1980), Poran (1985) and Burd (1986). Kwok (1987) used the plane strain finite element computer program CRISP (Britto and Gunn (1987)), which employs models of critical state soil mechanics, and incorporated special elements to model a reinforcement layer and the interfaces with the surrounding soil. The reinforcement was modelled by means of a line element with no bending stiffness and either a linear, or bi-linear axial force-extension relationship, while the interface elements permitted slip according to a Mohr-Coulomb criterion once the strength of the interface was exceeded. The embankment was modelled as an elastic-perfectly plastic material with a Mohr-Coulomb yield criterion and the foundation was modelled using the Modified Cam Clay model (Roscoe and Burland (1968)). The main conclusions of this work relevant to general reinforced two-layer soil systems, as described
in Hird and Kwok (1989) and (1990), are:

1) Sufficiently stiff and strong reinforcement can significantly reduce deformations within the subgrade. However an over-stiff reinforcement develops an excessively large tension without contributing significantly to an improvement in the load capacity. Therefore the benefit of increasing the reinforcement stiffness follows a pattern of strongly diminishing returns, (as substantiated in Chapter 8).

2) The reinforcement attracts shear stress, inducing tension, from both the fill and the subgrade, (as shown in Chapter 6).

The research work undertaken for this dissertation, on studies of reinforced and unreinforced two-layer soil systems, uses the finite element computer model OXFEM. This is a large displacement, large strain, finite element model that was initially developed by Burd (1986).

Burd (1986) describes a series of plane strain finite element predictions of the behaviour of a reinforced two-layer soil system consisting of a layer of fill compacted on top of a clay subgrade with rough, thin reinforcement placed at the interface. The numerical formulation has separate elements to model the soil and reinforcement and is specially designed to handle large deformations and large strains correctly (a full description of OXFEM is given in Chapter 3). The reinforcement was assumed perfectly rough, i.e. no slip or separation was allowed to occur, because the numerical model did not originally have the necessary interface elements incorporated. Later development of the computer program, as part of this research, involved installing special interface elements as described in Chapter 5.

In order to validate the mathematical model Burd (1986) conducted accurate back-analysis calculations of the eighteen experimental tests performed by Love (1984), (Section 2.3), using a consistent set of material properties. A number of general conclusions, relating to the behaviour of reinforced two-layer soil systems deforming in plane strain under the action of a single monotonic load, are derived from the results of these back-analysis
calculations:

1) The results clearly reproduce the important trend observed in the model tests that the reinforcement has a negligible effect on the load-penetration response at low surface deformations, but it becomes increasingly effective as displacements became appreciable.

2) For the cases studied by Burd (1986) the reinforcement was discovered to have little effect on the load spread action of the fill, as found for the parametric studies herein.

3) The reinforcement tended to reduce the shear stresses transmitted from the fill to the subgrade, i.e. the shear stress mechanism.

4) The numerical computations support the experimental findings of Love (1984) that the subgrade strength governs the load-displacement response of the system more strongly than the fill thickness.

Although Burd (1986) only considered the case of monotonic loading in plane strain, a development of that work is undertaken herein to deal with the additional case of axisymmetric monotonic loading (Chapters 4 and 5).

Finite element methods have become established as an important part of designing and analysing reinforced two-layer soil systems. However there are still some shortcomings associated with such computer modelling, for instance the practical requirement of sufficient computer capabilities and the need for accurate mathematical models. Additionally there are the limitations, shown in previous finite element studies, of needing to restrict the analyses to monotonic loading, or to a very low number of load cycles, rather than the realistic high-cyclic loading situation encountered for unpaved roads for example. There is also an apparent lack of extensive parametric studies to investigate the influences of various material properties.
2.5 Current Design Methods

There is clearly a need for analytical methods that may be used to predict the load-displacement characteristics of reinforced two-layer soil systems which practising engineers can use for design. Several design methods have therefore been proposed in recent years, based on simple analytical models of the reinforcement mechanism. The design models considered here are principally concerned with reinforced unpaved roads and are limited to the analysis of the road behaviour during a single static load application with adjustments made empirically for the cyclic loading case. This approach appears justified since Fannin (1986), (Section 2.3), found that the failure mechanisms under cyclic loading were the same as under monotonic loading.

2.5.1 Static Analyses

There have been many proposed design methods for reinforced unpaved roads, but only a few salient analyses for the static loading case are reviewed here, with an assessment of the particular limitations of each method given at the end of Section 2.5.1.

Giroud and Noiray (1981) describe a design method based on both the consideration of the ‘tensioned membrane effect’ and the subgrade confinement resulting from the curved nature of the geotextile after deformation, for the case of a plane strain reinforced unpaved road deforming under the action of a single application of a dual wheel load. The assumed mechanism of reinforcement is based on the link between footing displacement and membrane curvature. A pyramidal distribution of load through the aggregate layer is adopted and a constant value of \( \tan^{-1} 0.6 \) is taken for the load spread angle, irrespective of any other parameters. The deformed geotextile profile is assumed to follow a series of parabolae pinned at the initial plane of the fabric (see Figure 2.1), so that the average strain in the geotextile can be determined. The reinforcement tension is then evaluated as a function of the tensile stiffness and elongation of the membrane. In the reinforced case the reduction of pressure on the subgrade, due to the tension in the geotextile, is calculated and subtracted from the previously calculated pyramidal distribution. It is then suggested that this net load on the subgrade must be less than the ultimate bearing capacity of the subgrade which, due
to the additional confinement of the subgrade provided by the reinforcement, is given by the plastic bearing capacity factor, \((\pi + 2)\), in the reinforced case and the elastic limit bearing capacity, \(\pi\), in the unreinforced case.

From the results of the finite element parametric study conducted herein (Chapters 7-10), it is established that the shape of the deformed membrane is not a constant parabola, but is dependent upon several factors including the fill thickness, the reinforcement stiffness and the clay strength. Additionally, the improvement in load capacity is not directly proportional to the reinforcement stiffness and the load spread angle is found to be strongly dependent upon the clay strength.

**Giroud et al. (1984)** present a development of the Giroud and Noiray (1981) quasi-static design procedure by proposing two additional reinforcement mechanisms in an attempt to explain the improvement in capacity attained at very small rut depths. One, is that the presence of reinforcement at the base of the fill layer improves the load spread performance of the fill. The other mechanism is that the reinforcement improves the structural behaviour of the road by acting as a separator between the fill and the clay which helps prevent progressive deterioration of the base layer. These two additional mechanisms are based on purely empirical relationships. Additionally the authors consider that the tensioned membrane effect is only significant if the rut depth is \(\geq 150 \text{ mm}\) and is then accounted for by simply reducing the calculated granular layer thickness by 10%, otherwise it is neglected completely.

**Sellmeier et al. (1982)** consider the specific reinforcement mechanisms of ‘lateral restraint’ and ‘tensioned membrane action’. The design method initially assumes a bi-linear stress-deformation behaviour for the subgrade material with a separate elastic (Winkler spring type model) and plastic stress state analysis, although it is then concluded that the reinforcing effect of the membrane is negligible when the subsoil is in the elastic phase. A limit equilibrium equation is given relating the resultant of the combined traffic and aggregate loading on the upper side of the deformed geotextile and the subgrade reaction on the lower side, to the horizontal component of the tensile stress in the geotextile. It is suggested that the subgrade reaction stress is taken to be the failure stress estimated from a plastic analysis.
based upon the Brinch Hansen (1970) bearing capacity formula, however no indication is 
given of what bearing capacity factor to use. Additionally, a superfluous design specification 
is made concerning firm fixity at the road edges (Chapter 9).

De Groot et al. (1986) present a design method and guidelines for the application of 
geotextiles in unpaved, or high volume paved roads, based on the calculation method 
presented by Sellmeijer et al. (1982). To account for the lateral restraint effects the bearing 
capacity factor is assumed to be 3.0 without reinforcement and 5.0 with reinforcement, based 
purely on empirical data obtained from field tests. A requirement is also made for a minimum 
‘anchorage length’ of geotextile under the fill, over which the stress that has developed due 
to the strain in the heavily deformed part of the membrane will be reduced to zero by the 
friction on both sides of the geotextile (which is shown to be inaccurate in Chapter 9). An 
assessment of the separation function of a geotextile is presented in terms of its tear strength 
and puncture resistance. The problem of cyclic loading is also approached, as discussed in 
Section 2.5.2. Founded on these theories a step by step procedure is presented for designing 
the reinforced road.

Houlsby et al. (1989) propose a plane strain design method for reinforced unpaved 
roads where the benefit of reinforcement is considered only for the case of small surface 
displacements under a static monotonic load. Any improvement in the loading capacity of 
the reinforced road due to large deformations is excluded in this design approach. The valid 
justification of this is that the method is applicable to the design of hardstanding areas, such 
as car parks and working platforms, as well as unpaved roads, without the requirement for 
traffic channelling. The Houlsby et al. (1989) limit equilibrium method is based on the 
shear stress mechanism. In unreinforced roads the shear stresses at the fill base are sustained 
wholly by the subgrade, which has the detrimental effect of lowering the clay bearing 
capacity factor, $N_c$, to between $(2 + \pi)$ and $(1 + \pi/2)$, while in the reinforced road these 
shear stresses are assumed to be entirely removed by the membrane, leaving just a uniform 
vertical loading on the subgrade and allowing the maximum bearing capacity of $N_c = (2 + \pi)$.

The Houlsby et al. (1989) analysis combines the unknown values of the bearing 
capacity of the clay subgrade and the horizontal stresses in the fill, which are linked by the
shear stresses developed at the unreinforced fill-clay interface. It is assumed that the fill beneath the footing is in a state of active failure and that outside of the loaded area it is in a general state of passive failure. The active and passive earth pressure coefficients give the corresponding lateral stresses in the fill. Through the equation of horizontal equilibrium the shear stresses acting at the base of the fill layer are evaluated, which gives the maximum value of the bearing capacity factor, and consequently the largest footing load can be calculated for a given fill depth.

Two of the assumptions made within the method are that, (1) the vertical load from the footing adopts an uniform pyramidal load distribution at an angle $\beta$ through the fill layer and, (2) that an inward acting frictional force exists between the footing base and the fill layer surface, given by an angle of friction $\delta_f$. The horizontal footing friction force is significant in the calculation of the maximum unreinforced footing load and the shear stresses at the fill base.

The reinforcement tension is estimated to be equal to the product of the total outward shear stress within the load spread area and the distance from the footing centre-line for which it acts. To carry this load there is an implicit requirement for a minimum value of both reinforcement stiffness and reinforcement length, although anchorage of the membrane is considered to be unnecessary since the reinforcement is under tension due to the outward acting shear stresses, even at small footing displacements. The possibility of failure within the fill is determined through a conventional bearing capacity investigation using the approximation of the self-weight bearing capacity factor, $N_r$, suggested by Vesic (1975):

\[
N_r = 2 \left( \exp(\pi \tan \phi) \tan^2(45^\circ + \phi/2) + 1 \right) \tan \phi
\]  

(2.1)

Houlsby and Jewell (1990) present an extended analysis of the Houlsby et al. (1989) theory for the case of axisymmetry, since in general a wheel loading approximates more closely to an axisymmetric rather than a plane strain style of loading. The analysis considers a static monotonic load, where the rut depth is again required to remain small. The authors suggest that such an analysis is justified since road designers generally want an assurance that their road will perform adequately at low surface deformation thus avoiding secondary loading, from water accumulation for example. The principle differences between the plane
strain and axisymmetric analyses are that, (1) the three-dimensional load spread gives a more rapid reduction in vertical stress within the fill, (2) the bearing capacity factors are somewhat higher (i.e. the full bearing capacity factor is 5.694 in axisymmetry, as opposed to \((\pi + 2) \approx 5.14\) in plane strain), and (3) the passive pressures from the fill have greater influence. All three effects tend to increase the allowable loading on the pavement in comparison with the plane strain analysis. Design charts are developed for the unreinforced and reinforced cases.

Sellmeijer (1990) calculates the required strength of the geotextile needed to ensure that the membrane and the aggregate are both in a state of equilibrium. The method assumes that vertical equilibrium is established as indicated in Figure 2.3 (a), where \(\sigma_v\) is the limiting stress available from the subgrade, and hence the width of areas 2, 3 and 4 may be calculated. The resulting vertical deformation of the geotextile depends only on the shear modulus of the aggregate and through strain compatibility between the geotextile and aggregate the stiffness of the reinforcement may be computed. It is assumed that due to lateral restraint the subbase layer acts as a continuous slab and that it is reinforced by the geotextile in a similar way to a steel bar reinforcing concrete. Additionally, the mode of deformation experienced by the aggregate layer is taken to be one of shearing, rather than bending, Figure 2.3 (b). The other function of the geotextile is to maintain the integrity of the individual soil layers. The fill is considered to behave in an elasto-plastic manner and the mobilised friction angle, which is limited to values below the internal fill friction angle, is used in the analysis to control the degree of strain at the fill-reinforcement interface. An important parameter in the analysis is the friction angle between the geotextile and aggregate, which must be of an optimum value to ensure that the aggregate displays a slab effect. It is assumed that the reinforcement does not directly reduce the subgrade stress, but that the internal outward deformation of the aggregate is restricted by virtue of the aggregate-geotextile friction, thus improving the aggregate shear modulus and consequently reducing any rutting.

Developing a comprehensive analytical design method for reinforced two-layer soil systems is undoubtedly complex and therefore necessitates that certain simplifications and modifications are made to the actual problem. One particular difficulty is accounting for both the large and small displacement mechanisms operating in the system. For example,
2.3 (a) Vertical Equilibrium
(After Sellmeijer (1990))

2.3 (b) Modes of Aggregate Layer
Deformation

Figure 2.3:

reinforced unpaved roads will ordinarily deform quite noticeably during service, however, it is found that there are significant improvements in the load bearing capacity of a reinforced two-layer soil system, compared to an unreinforced, even at very small rut depths, Love et al. (1987). Therefore, the design method needs to be based on an understanding of the mechanics involved at both small and large displacements.

The design procedures discussed in this Section 2.5.1 each have certain drawbacks. Giroud and Noiray (1981) base their design on the limiting, or inaccurate, assumptions that large displacements occur at the surface, that the deformed geotextile profile follows a series of parabolas pinned at the initial plane of the fabric and that the improvement in load capacity is linear with the reinforcement stiffness. Although Giroud et al. (1984) further develop the rudimentary theory of Giroud and Noiray (1981), such that reinforcement benefit is estimated for small rut depths and a load distribution improvement is calculated, there is still a simplistic analysis of the subgrade bearing capacity for an unreinforced road which does not consider any shear stress effects at the fill base. Sellmeijer et al. (1982) makes the erroneous assumptions that anchorage of the geotextile is necessary and that the reinforcement is sufficiently frictional so that slip cannot occur, in order to calculate the strain in the reinforcement due to the surface rutting. De Groot et al. (1986) base their design merely
on empirical evaluation of the subgrade bearing capacity, rather than any detailed mathematical investigation. The design method by Sellmeijer (1990) is inconsistent by assuming an elastic fill layer and plastic subgrade, and restrictive by considering the fill layer of a reinforced unpaved road as a stiff slab, whereas it has been found through this research that in fact the aggregate layer develops plastic yield (see Figure 6.5), before failure occurs in the clay subgrade.

Despite more sophisticated methods of analysis by Houlsby et al. (1989) and Houlsby and Jewell (1990) which calculate the lower varying bearing capacities obtainable for unreinforced cases, as opposed to the constant ultimate values used in fully reinforced cases, the methods do not consider the improved wheel loading that will undoubtedly be obtained due to the tensioned membrane mechanism. Additionally, there is no consideration of the exact reinforcement stiffness, only a prerequisite that it is sufficiently stiff to create a ‘fully’ reinforced condition and, moreover, it is assumed incorrectly that active fill failure conditions prevail directly beneath the footing (see Section 6.5.6). Although these analytical design methods are limited to purely small displacement, they are considered to be the most progressive and complete design methods for unpaved roads to date (Dawson and Brown (1992)) and they will therefore be used throughout the parametric investigation, carried out in Chapters 6 - 10, for a comparative study between the analytical solutions they provide and the finite element results obtained.

All these design methods are primarily concerned with the static loading case, but obviously a reinforced road is generally subjected to cyclic loading.

2.5.2 Cyclic Analyses

For most road applications the loads will be repetitive, which requires a modified approach in the design analysis.

The analytical design methods proposed by Giroud and Noiray (1981) and Giroud et al. (1984) are both quasi-static analyses which are converted to a cyclic loading design by applying an empirical formula relating the aggregate layer thickness to the amount of traffic
and to the soil properties for unreinforced unpaved roads. A set of empirically derived design charts are presented for the adjustment of the fill layer necessary to compensate for progressive deterioration of the clay subgrade due to repeated loading.

The De Groot et al. (1986) design method allows for cyclic loading conditions by deriving an equivalent static failure load, \( P_s \), from an empirical fatigue relationship (equation (2.2)) derived from full-scale trafficking trials:

\[
P_s = P_n N^{0.161}
\]  

(2.2)

where \( P_n \) is the allowable axle load for \( N \) applications of the load.

Milligan et al. (1989), the companion paper of Houlsby et al. (1989), estimates the effect of repeated loading on a reinforced and unreinforced unpaved road by comparing the experimental cyclic loading data of Fannin (1986) (Section 2.3) with similar published studies by Delmas et al. (1986), De Groot et al. (1986) and Hammit (1970). It is suggested by Milligan et al. (1989) that the fatigue curve given by the De Groot et al. (1986) analysis (equation (2.2)) is currently the best available method for designing a reinforced unpaved road and that a similar fatigue relationship presented graphically by Hammit (1970) should be used for unreinforced roads. Milligan et al. (1989) provides an overall design procedure and worked examples of reinforced unpaved roads using the Houlsby et al. (1989) theory.

A paper by Davies and Bridle (1990) predicts the gradually increasing permanent deformation of a reinforced road due to repeated monotonic loading, by using a method based on the theorem of the minimum potential energy. The sum of the potential energy of the system and the work done by the applied load is equated to the sum of the strain energies in the subgrade, subbase and geogrid. Modelling of the permanent deformations remaining after a loading cycle, is achieved by choice of the properties of the subgrade. The cumulative deformation that occurs with repeated loading is modelled by variation of the material parameters of the subgrade. Although the analysis is compared to a restricted number of experimental model tests, promising results were obtained regarding its validity.

Despite the crudity of using empirical relationships established from field trials it is generally accepted at present that such methods are valid. This thesis, however, is concerned only with the static loading case and further discussion of cyclic loading is extraneous.
CHAPTER 3:

THE FINITE ELEMENT MODEL

3.1 Introduction

The research described in this thesis was carried out using the finite element program OXFEM, which is fully described in Burd (1986) and Yu (1990). This chapter provides a brief overview of aspects of the model that are not described in detail later in this thesis.

The program OXFEM is specifically designed to conduct large displacement, large strain calculations for both the plane strain and axisymmetric analyses of geotechnical structures. The specific application considered here is that of a two-layer soil system, with and without a layer of geosynthetic reinforcement placed between the soil layers, subjected to a single monotonic load.

3.2 Large Displacement Analyses

Some of the reinforcement mechanisms in a two-layer soil system only begin to operate at large surface displacements, e.g. the tensioned membrane effect, as discussed in Section 2.2. The numerical analysis, therefore, needs to be able to handle accurately any geometric non-linearity that may occur during deformation, i.e. it needs to be a large strain formulation. In OXFEM an Updated Lagrangian description of the deformation is used (i.e. the reference state is updated as the calculation proceeds, hence all the strain rates are related to the current mesh configuration), with extra non-linear terms included in the finite element stiffness
equations to account for geometric non-linearities and the addition of the Jaumann stress rate in the constitutive equations to account for large rotation effects. These large displacement characteristics are discussed more fully in the following sections.

3.2.1 Description of Kinematics

By using a Lagrangian reference system in continuum mechanics the deformation of a body subjected to large displacements is uniquely defined by a vector equation of the form:

\[ x_i = x_i (a_i, t) \]  \hspace{1cm} (3.1)

where the reference position (at time \( t = 0 \)) of a particular point within the material is described by the co-ordinate vector, \( a_i \), relative to some fixed origin. After a finite time interval (\( t = T \)) the same material point will have moved to a new position described by the vector, \( x_i \), in the same reference frame. This Updated Lagrangian description of kinematics, where the reference position is updated as the calculation proceeds, has been used for all the OXFEM finite element calculations discussed herein.

3.2.2 Objective Stress Rates

The objective stress rate adopted in the OXFEM large strain calculations is the Jaumann stress rate (Jaumann (1911)). The requirement of ‘objectivity’ is that when a material experiences rigid body motion, or rotation, the stress rate used in the constitutive equations must be zero.

A unique advantage of the Jaumann definition is that a zero stress rate represents a stationary behaviour of the stress invariants and similarly of the yield function for an isotropic material. However, the Jaumann stress rate does have the restriction that for problems involving large shear strains, i.e. greater than unity, then physically unrealistic solutions may be given. Despite this limitation the original Jaumann stress rate has been used in this study since the shear strains are not expected to be excessively large.
3.3 Continuum Elements to Model Soil

In order to ascertain whether a particular element assemblage is suitable for accurate elasto-plastic analysis Nagtegaal et al. (1974) proposed a criterion where it is necessary to evaluate the number of degrees-of-freedom per constraint for the limiting case of a very fine mesh. If this ratio is less than one, convergence to the true limit load will generally not be obtained.

Sloan (1981) applies this criterion to a large variety of triangular and serendipity elements for both the plane strain and axisymmetric cases. Sloan concludes that for plane strain problems of undrained soil behaviour, where the constant volume condition must be maintained, (i.e. material is incompressible) the majority of triangular and serendipity elements in common usage are suitable for predicting collapse loads accurately. The noticeable exception being the plane strain four-noded quadrilateral element. However, in the case of axisymmetric problems Sloan shows that only the fifteen-noded cubic strain triangle, or the twenty five-noded quadrilateral element, (or higher orders if desired) give a ratio of the number of degrees-of-freedom per constraint greater than one.

In all the finite element calculations reported in this thesis isoparametric triangular continuum elements are used to represent the soil medium, i.e. the equations describing the shape of the element boundaries are of the same order as those describing the variation of the nodal unknown (e.g. displacements) across the element. For the plane strain analyses six-noded elements are implemented, while fifteen-noded elements are used for the axisymmetric case, Figure 3.1, since these are the minimum number of nodes necessary to predict collapse loads accurately.

The specific finite element equations for the six-noded plane strain triangular continuum element are given in Burd (1986) and for the fifteen-noded axisymmetric continuum element see Teh (1987).
Chapter 3: The Finite Element Model

Six-noded Plane Strain

Fifteen-noded Axisymmetric

Figure 3.1: Triangular Continuum Elements

3.4 Global Solution Scheme

The governing equations in finite element analysis are derived from the principle of virtual work, which states that the external applied loads, \( P \), must be in equilibrium with the current internal stresses, \( \sigma \). This is usually expressed in matrix form (Zienkiewicz 1977) as:

\[
P = \int_E [B]^T \sigma \, dV
\]  
(3.2)

where the integration is performed over the element and the matrix \([B]\) relates the nodal velocities vector, \( U \), to the strain rate vector, \( \dot{\varepsilon} \), as follows:

\[
\dot{\varepsilon} = [B] U
\]  
(3.3)

where the superior dot denotes differentiation with respect to time.
The matrix \([B]\) depends on the strain definitions and element shape functions used to specify the displacement field throughout the element. By introducing the elasto-plastic material stiffness matrix, \([D]\)\(^{EP}\), which defines the relevant stress-strain relationship such that:

\[
\dot{\varepsilon} = [D]^{EP} \dot{\varepsilon}
\]  

(3.4)

equation (3.2) can be differentiated and written in terms of the nodal force rates and nodal velocities as:

\[
\dot{P} = [K] U
\]  

(3.5)

where \([K]\) is the tangential incremental stiffness matrix and is expressed as:

\[
\]  

(3.6)

The specific incremental stiffness matrix for the plane strain case is defined by Burd (1986) as:

\[
[K]_{PS} = \int \int_E \{ [B]^T ([D]^{EP} + [R]) [B] \, \text{det}[J] + [C]_{PS} \} \, d\alpha \, d\beta
\]  

(3.7)

and the axisymmetric incremental stiffness matrix is defined by Teh (1987) as:

\[
[K]_{Asi} = 2\pi \int \int_E \{ [B]^T ([D]^{EP} + [R]) [B] r \, \text{det}[J] + [C]_{Asi} \} \, d\alpha \, d\beta
\]  

(3.8)

where \(\alpha\) and \(\beta\) are the reference co-ordinates, \(r\) is the radius, \([J]\) is the Jacobian of the transformation from global to reference co-ordinates, the matrix \([R]\) accounts for the effects of element rotation and the matrices \([C]_{PS}\) and \([C]_{Asi}\) are interpreted as the distortion rate matrices. The stiffness matrices, \([K]_{PS}\) and \([K]_{Asi}\), therefore contain the conventional small displacement terms as well as an additional term, \([C]\), which accounts for the rate of change.
of element geometry and is different for the plane strain and axisymmetric cases. The plane strain non-symmetrical distortion rate matrix is detailed by Burd (1986) and is obtained by expanding the following expression on a term-by-term basis.

\[
[C]_{ps} U = \{ [B]^T \dot{det}[J] \} \sigma
\]  

(3.9)

The axisymmetric non-symmetrical distortion rate matrix is obtained by Teh (1987) through combining two of the terms in the resulting integrand, i.e.:

\[
[C]_{Asi} U = \{ [B]^T \dot{det}[J] \} \sigma r + [B]^T \sigma r \dot{det}[J]
\]  

(3.10)

In finite element problems where material and/or geometric non-linearities exist, as in the large displacement two-layer soil system analyses reported in this thesis, the resulting equations are non-linear and must be solved using an appropriate algorithm. The chosen incremental solution procedure implemented in OXFEM is described by Sloan (1981) as the ‘modified Euler scheme’. A schematic representation of this scheme for a one degree-of-freedom system is presented in Figure 3.2. The exact load-displacement curve is approximated by a series of straight lines having a slope equal to that of the exact curve at the start of the increment. At the end of each increment of load, or displacement, the nodal forces corresponding to the equilibrium imbalance are calculated and equilibrium is restored by applying these out-of-balance loads in the opposite direction during the next load increment.

In a displacement controlled calculation the nodal force, \( P_i \), in equilibrium with the prescribed nodal displacement, \( U_i \), at the start of the \( i^{th} \) load step is used in equation (3.5) to obtain the incremental nodal force, \( \Delta F_{i+1} \). The individual element stiffness matrices are calculated and then assembled and inverted using a Frontal Solution algorithm.

The Frontal Solution, as outlined by Bathe (1982) and implemented in OXFEM by Burd (1986), is in principle a Gauss elimination procedure. The stiffness matrix of equation (3.6) is invariably calculated using a Gaussian quadrature scheme. The relevant order of the Gaussian quadrature scheme must be determined from the degree of the polynomial to be evaluated (see Section 4.2.2). However, the accuracy of the integration is affected when
large element distortion occurs and the calculations cannot be considered exact for a large displacement finite element analysis. For the six-noded plane strain elements a three-point Gauss rule is used, while the fifteen-noded axisymmetric elements use a sixteen-point Gauss rule (Britto and Gunn (1987)).

The set of Gauss point stress increments, $\Delta \sigma$, corresponding to the set of incremental nodal displacements, $\Delta U_{i+1}$, are then calculated using an integration of the form:

$$\Delta \sigma = \int_{1}^{i+\Delta t} [D]^{op} \dot{\varepsilon} \, dt$$

where the integration is performed over the time step $\Delta t$. A complete description of the ‘stress update’ calculations is given in Burd (1986) and Teh (1987) for the plane strain and axisymmetric continuum elements and in Chapters 4 and 5 for the new axisymmetric membrane and interface elements, respectively.
At the end of each step the nodal loads, $P_i$, equivalent to the updated stresses are calculated from the virtual work equation (3.2). A set of unbalanced nodal loads, $\Delta R_i$, are then determined and applied as an equilibrium correction during the next displacement increment.

3.5 Constitutive Behaviour

The clay subgrade and the granular fill materials of a two-layer soil system require different constitutive laws in order to numerically model their behaviour appropriately. Linear-elastic perfectly-plastic models of behaviour are used in OXFEM to simulate the two soil materials.

![Figure 3.3: Comparison of Yield Surfaces](image)

(After Burd (1986))

It is assumed that the loading on the subgrade is sufficiently rapid to give an undrained response and consequently is modelled using a 'total stress’ approach. The von Mises plasticity yield criterion (Calladine (1985)), shown in Figure 3.3 for a section through the yield loci in the $\pi$ plane, is used to model the cohesive clay. The plastic strain rates are derived from a fully associated flow rule which gives zero plastic dilation rate and is therefore suitable to the modelling of undrained clay behaviour, as discussed by Wroth and Houlsby.
(1985). A fully associated flow rule requires that the plastic potential and the von Mises yield functions are exactly identical when plotted together using consistent axes and the plastic strains are then normal to the yield surface.

The fill is assumed sufficiently permeable not to allow pore pressures to occur during the loading, hence total and effective stresses are equal. It is represented as a frictional material using a non-associated linear elastic perfectly-plastic model based on the Matsuoka yield criterion (Matsuoka (1976)), illustrated in Figure 3.3. By using a non-associated flow rule the dilation characteristics of the soil are specified independently from the friction angle. A parameter termed the degree-of-association, $\gamma_a$, is introduced, which when set to unity the model of behaviour reduces to the full association case and when $\gamma_a$ is zero the dilation rate is zero. The material behaviour for the case when the dilation rate lies between these two extremes is obtained by taking a weighted average of the plastic strain rates that give full and zero dilation.

Alternative yield functions such as the Tresca and Mohr-Coulomb models, (shown in Figure 3.3) commonly used to represent cohesive and frictional soils respectively, contain discontinuous yield surfaces with edges at which the yield function is not differentiable. In axisymmetric loading conditions, particularly, these singularities can have an important effect (Sloan (1981)). However, the von Mises and Matsuoka yield functions have the advantage that they are everywhere differentiable (except at the origin for the Matsuoka model). A full description of the approach used in implementing these formulations is given in Burd and Houlsby (1989) and Burd (1986).
CHAPTER 4:

FORMULATION OF THE MEMBRANE ELEMENTS

4.1 Introduction

This chapter is devoted to the description of the formulation of an axisymmetric membrane element possessing a finite axial stiffness, but a zero bending stiffness. The formulation intrinsically follows the approach described by Burd and Houlsby (1986) for a plane strain membrane element. However, that formulation needs to be modified for the axisymmetric case because of the difference in strain arrangement.

This axisymmetric membrane element effectively models a continuous sheet of material. It assumes that the membrane possesses isotropic material properties, where the material stiffness is the same for the tangential and circumferential directions.

The finite element equations are based on an Updated Lagrangian description of deformation, in a similar manner to the continuum element formulation described in Chapter 3. The Updated Lagrangian description of deformation lends itself well to the use of the Hencky strain rate to define the material behaviour of the membrane. Therefore, for this particular formulation, the material is assumed to be linearly elastic with respect to Hencky strain. In general, the stiffness equations derived using this approach will be different from those derived for a material that is assumed linearly elastic with respect to some alternative strain definition.
The proceeding sections include a detailed account of the finite element formulation for the new axisymmetric membrane element, followed by a brief synopsis of how this formulation differs to that of the established plane strain membrane element given by Burd and Houlsby (1986) and concluding with a set of rigorous test analyses.

4.2 Axisymmetric Membrane Element Formulation

The large strain finite element formulation of the new axisymmetric membrane element is based on an isoparametric approach, i.e. the shape functions defining the element geometry and the nodal displacements are the same. The formulation takes account of the effects of large strains and rotations by calculating the incremental strains generated at each step of the analysis and continuously updating these, along with the associated stresses, throughout the calculation.

In an axisymmetric problem any tangential displacement of the membrane naturally creates a tangential strain and thus a tangential stress \( \sigma_t \), but it additionally induces a strain in the circumferential direction and therefore an associated circumferential stress, \( \sigma_\theta \), exists, as shown in Figure 4.1. For this particular formulation the membrane is considered to be of an infinitesimal initial thickness, \( t_0 \), and consequently the normal stress, \( \sigma_r \), acting in the membrane will be comparatively very much smaller in relation to the in-plane stresses and can therefore be neglected. The variation of the membrane thickness is accounted for at each increment of load. The one-dimensional membrane element is also assumed to have zero bending moment and therefore no shear stress component needs to be considered within the element. Consequently it is only the tangential and circumferential stresses that are calculated in this new axisymmetric membrane formulation.

The axisymmetric formulation described in this chapter is applicable to membrane elements of arbitrary order and is therefore presented in a general form. The specific equations for the case of the five-noded element are presented in Appendix 4A.
\[
\sigma_n \ll \sigma_r, \sigma_\theta
\]

Figure 4.1: Stresses in the Axisymmetric Membrane

4.2.1 Strain Definitions

The tangential and circumferential strains are individually defined in terms of the Hencky strain rate. Consider the tangential displacement of an infinitesimal portion of the membrane during an infinitesimal time step \( dt \), as shown in Figure 4.2. Cylindrical co-ordinates are denoted by \( r, z \) and \( u, v \) are the velocities in the \( r, z \) directions respectively.

Figure 4.2: Displacements in a Membrane Element
The fractional increase in the length of the portion is:

\[
\frac{\left[ (dr + du dt)^2 + (dz + dv dt)^2 \right]^{1/2} - \left[ dr^2 + dz^2 \right]^{1/2}}{\left[ dr^2 + dz^2 \right]^{1/2}} = \frac{dr du + dz dv}{dr^2 + dz^2} \frac{dt}{dt}
\]

(4.1)

where higher order displacement terms are ignored. The tangential Hencky strain rate, \( \dot{e}_t \), can therefore be written:

\[
\dot{e}_t = \frac{de_t}{dt} = \frac{dr du + dz dv}{dr^2 + dz^2}
\]

(4.2)

where the superior dot denotes differentiation with respect to time.

The circumferential Hencky strain rate, \( \dot{e}_\theta \), is automatically given by the scalar value:

\[
\dot{e}_\theta = \frac{u}{r}
\]

(4.3)

(See Slater (1977), or Timoshenko and Goodier (1951)).

In order to develop the relationships between strain rates and nodal velocities, an axisymmetric isoparametric membrane element with \( N \) nodes is considered. Each node has two degrees of freedom. The global cylindrical co-ordinates of a point in the element are \( r, z \) and these are mapped onto a single reference co-ordinate \( \alpha \), as shown in Figure 4.3.

The vector of global co-ordinates of an arbitrary point in the element, \( r^T = [ r \quad z ] \), is related to the vector of global nodal co-ordinates \( R \), by:

\[
r = [N] R
\]

(4.4)

where \([N]\), the shape function matrix, is a function of \( \alpha \). In a similar way the velocity vector of the same arbitrary point, \( u^T = [ u \quad v ] \), is related to the nodal velocity vector \( U \) by:

\[
u = [N] U
\]

(4.5)
Chapter 4: Formulation of the Membrane Elements

![Diagram of a mapping for an axisymmetric membrane element]

**Figure 4.3: Mapping for an Axisymmetric Membrane Element**

Differentiating these two equations (4.4) and (4.5) with respect to \( \alpha \) and denoting this with a prime, gives:

\[
\begin{align*}
r' &= [N'] R \\
u' &= [N'] U
\end{align*}
\]  
(4.6)  
(4.7)

The tangential Hencky strain rate equation (4.2) can be written in terms of the mapped co-ordinate vector \( r' \) and the mapped velocity vector \( u' \), as:

\[
\begin{align*}
\dot{\varepsilon}_t &= \frac{dr}{d\alpha} \frac{du}{d\alpha} + \frac{dz}{d\alpha} \frac{dv}{d\alpha} \\
&\quad + \frac{dr}{d\alpha} \frac{dz}{d\alpha} \\
&= \left( \frac{r'}{r} \right)^T u' \\
&\quad \left( \frac{r'}{r} \right)^T \frac{r'}{r}
\end{align*}
\]  
(4.8)
which can be re-arranged, by substituting equations (4.6) and (4.7), to:

\[
\dot{\varepsilon}_r = \frac{R^T [N']^T [N'] U}{R^T [N']^T [N'] R} \tag{4.9}
\]

The denominator of the expression given in equation (4.9) can be denoted by \( J^2 \), where \( J \) represents the Jacobian of the transformation from parent to reference co-ordinates and is given by:

\[
J^2 = \left( \frac{dr}{d\alpha} \right)^2 + \left( \frac{dz}{d\alpha} \right)^2 \tag{4.10}
\]

Differentiating equation (4.10) with respect to time gives:

\[
2 \frac{dJ}{dt} = 2 \left( \frac{dr}{d\alpha} \right) \left( \frac{du}{d\alpha} \right) + 2 \left( \frac{dz}{d\alpha} \right) \left( \frac{dv}{d\alpha} \right)
\]

and by substituting equations (4.6) and (4.7), this re-arranges to:

\[
J = \frac{R^T [N']^T [N'] U}{R} \tag{4.12}
\]

Thus, equation (4.9) can be written in the simplified form:

\[
\dot{\varepsilon}_r = \frac{\dot{J}}{J} \tag{4.13}
\]

Rewriting the circumferential strain rate equation (4.3) in terms of the global co-ordinate vector \( \vec{r} \) and the velocity vector \( \vec{u} \), gives:

\[
\dot{\varepsilon}_\theta = \frac{\dot{r}}{r} = \frac{k [N] U}{k [N] R} \tag{4.14}
\]
where the vector constant $k = [1 \ 0]$ is introduced so that the radius $r$ is equal to the scalar value of $k \ [N] \ R$

By using the tangential and circumferential strain rates defined by equations (4.9) and (4.14) respectively, the strain rate vector, $\dot{\epsilon}$, may be written as:

$$\dot{\epsilon} = \begin{bmatrix} \dot{\epsilon}_t \\ \dot{\epsilon}_\theta \end{bmatrix} = \begin{bmatrix} R^T \ [N']^T \ [N'] \\ \frac{J^2}{k \ [N]} \\ \frac{k \ [N] R}{k \ [N]} \end{bmatrix} \ U = [B] \ U \tag{4.15}$$

where the $[B]$ matrix relates the nodal velocity vector $U$ to the strain rate vector $\dot{\epsilon}$.

4.2.2 Formulation of the Finite Element Equations

By the principle of virtual work:

$$U^*^T \ P = \int_E \dot{\epsilon}^*^T \ \sigma \ dV \tag{4.16}$$

Where $U^*$ is a set of virtual nodal velocities and $\dot{\epsilon}^*$ is the vector of the compatible tangential and circumferential strain rates. The vector of nodal forces, denoted as $P$, is in equilibrium with the stress vector $\sigma^T = [\sigma_t \ \sigma_\theta]$, and the integration is performed over each individual element.

Substituting the strain rate vector, equation (4.15), into equation (4.16) gives:

$$U^*^T \ P = \int_E U^*^T \ [B]^T \ \sigma \ dV \tag{4.17}$$
The elemental volume $dV$ may be written as:

$$dV = 2\pi r J T \ d\alpha$$  \hspace{1cm} (4.18)

where $T$ is the instantaneous thickness of the membrane element and $J \ d\alpha$ can be interpreted as representing the tangential length of an infinitesimal element within the membrane. Since the virtual work expression must hold for arbitrary nodal velocities, it follows that equation (4.17) can be re-arranged to give the nodal forces as:

$$P = 2\pi \int_E [B]^T \sigma r J T \ d\alpha$$  \hspace{1cm} (4.19)

Equation (4.19) may be differentiated with respect to time to give an expression for the nodal force rate:

$$\dot{P} = 2\pi \int_E \left\{ [B]^T \dot{\sigma} r J T + [\dot{B}]^T \sigma r J T + [B]^T \sigma \dot{r} J T + [B]^T \sigma r J \dot{T} \right\} \ d\alpha$$  \hspace{1cm} (4.20)

which can be simplified to:

$$\dot{P} = [K] \dot{U}$$  \hspace{1cm} (4.21)

where $[K]$ is the incremental stiffness matrix given by:

$$[K] = 2\pi \int_E \left\{ [B]^T [D] [B] r J T + [C] \right\} \ d\alpha$$  \hspace{1cm} (4.22)

The material stiffness matrix, $[D]$, in equation (4.22) relates the stress rate vector $\dot{\sigma}$ to the strain rate vector $\dot{\varepsilon}$ by:

$$\dot{\sigma} = [D] \dot{\varepsilon}$$  \hspace{1cm} (4.23)
Note that it is possible to express the \([K]\) matrix in a variety of forms. The form given in equation (4.22) is chosen to be consistent with the axisymmetric continuum formulation (Teh (1987)), equation (3.8). The first term in the integrand of the incremental stiffness matrix is the conventional small displacement term. The second term, \([C]\), contains additional terms that need to be included to represent the effect of geometry changes on the nodal force rate. This matrix is termed the distortion rate matrix (Burd (1986)).

The distortion rate matrix is given by:

\[
[C] \dot{U} = [\hat{B}]^T \sigma r J T + [B]^T \sigma r \dot{J} T + [B]^T \sigma r J \dot{T} + [B]^T \sigma r J \dot{T} \quad (4.24)
\]

The four terms in the distortion rate matrix each represent specific components of the nodal force rate equation. The first term gives the rate of change of element geometry and the three remaining terms account for the rate of change of the radius \((r)\), the Jacobian \((J)\) and the thickness \((T)\). It is not possible to find \([C]\) directly by matrix manipulation, instead it is considered in two parts and rewritten as:

\[
[C] \dot{U} = [C_1] \dot{U} r J T + [B]^T \sigma r J T \left( \frac{\dot{r}}{r} + \frac{\dot{J}}{J} + \frac{\dot{T}}{T} \right) \quad (4.25)
\]

where:

\[
[C_1] \dot{U} = [\hat{B}]^T \sigma \quad (4.26)
\]

The first term of equation (4.25) is evaluated by expanding it on a term-by-term basis and the second term is obtained by matrix manipulation. The resulting \([C_1]\) matrix for the specific case of a five-noded axisymmetric membrane element is defined in Appendix 4A. The substitution procedure used in equation (4.26) was initially developed by Burd (1986) for the plane strain continuum element formulation. It is noteworthy that for the plane strain membrane element, however, this matrix solving was avoided by algebraic manipulation of the matrix equations, as discussed in Section 4.3.
Since the stresses normal to the membrane are neglected, the proportional thickness change of the membrane can be written in terms of the tangential and circumferential strain rates:

\[
\frac{\dot{T}}{T} = -\frac{v}{1-v} (\dot{\varepsilon}_t + \dot{\varepsilon}_\theta)
\]  

(4.27)

The distortion rate matrix, equation (4.25), may therefore be rewritten using equations (4.12), (4.13), (4.14) and (4.27) to give:

\[
[C] = \left\{ [C_1] + [B]^T \sigma \left( \frac{r}{R} \left[ N^T \frac{N'}{R} \frac{N'}{J^2} \frac{R}{J^2} \frac{k}{r} \right] \right) \right\} J T
\]  

(4.28)

All of the reinforced axisymmetric computations described in this thesis were carried out using a five-noded axisymmetric membrane element, which is compatible with the fifteen-noded axisymmetric continuum element, i.e. the same order of polynomial is used for the displacement interpolation. Additionally, a seven-point Gaussian quadrature scheme was used to evaluate the incremental stiffness matrix \([K]\) of equation (4.22). To obtain an exact integration the required number of Gauss points is actually determined by the degree of the polynomial to be evaluated. In general, a polynomial of degree \(2n - 1\) is integrated exactly by an \(n\)-point Gauss quadrature.

The polynomial order of \([K]\) can actually be determined from the nodal force vector equation (4.19), since it is by simply differentiating this expression with respect to time that the incremental stiffness matrix is obtained. In a small displacement problem the membrane element does not undergo any severe distortions and, assuming it is initially set up straight and that the nodes are equally spaced, the Jacobian of the transformation from global to reference co-ordinates, \(J\), is a constant. Similarly, the global co-ordinates vector of any arbitrary point in the element, \(r'\), is also a constant. Consequently, the integrand in equation
(4.19) can be shown from Appendix 4A to be a polynomial of the tenth degree. Therefore, a six-point Gauss quadrature would be sufficient to integrate \([K]\) exactly for the small displacement problem.

However, once the element becomes distorted due to large displacements the determinant of the Jacobian, \(J\), no longer remains a constant and the integrand of equation (4.22) becomes a ratio of polynomials, which corresponds to a polynomial of infinite degree. Therefore a seven-point Gaussian integration rule has been employed to evaluate the element stiffness matrix, since the use of any higher order routines will make very little difference.

4.2.3 Constitutive Equations

The axisymmetric membrane is modelled as a linear elastic isotropic material with a single value of Young’s modulus, \(E\), a Poisson’s ratio, \(\nu\), and zero bending stiffness, since geosynthetics generally cannot sustain any significantly large compressions without buckling. In practice many different geosynthetic reinforcement materials are available, some of which are isotropic, for example spunbonded polypropylene geotextiles, while others are anisotropic, like the geogrids which possess different stiffnesses in the transverse and longitudinal directions. Although this axisymmetric membrane element is developed for the case of isotropy, an anisotropic membrane could be developed by modifying the material stiffness matrix appropriately.

Since the membrane has zero bending stiffness each of the tangential and circumferential stresses can independently be zero, i.e. buckle. In order to include these features in the formulation the following procedures are adopted for the four possible cases of constitutive behaviour.

**Case (1):** \(\sigma_t > 0\) and \(\sigma_\theta > 0\)

From Hooke’s Laws, assuming isotropic Young’s modulus:-

\[
\dot{\varepsilon}_t = \frac{\dot{\sigma}_t}{E} - \nu \frac{\dot{\sigma}_\theta}{E}
\]  

(4.29)
\[ \dot{\varepsilon}_\theta = \frac{\dot{\sigma}_\theta}{E} - \nu \frac{\dot{\sigma}_r}{E} \]  

(4.30)

These equations may be re-arranged in the form of equation (4.23), such that the material stiffness matrix may be shown to be:-

\[
[D] = \begin{bmatrix}
\frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} \\
\frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2}
\end{bmatrix}
\]  

(4.31)

**Case (2):** \( \sigma_r > 0 \) and \( \sigma_\theta = 0 \)

In this case the membrane is buckled in the circumferential direction. The value of \( \sigma_\theta \) is set to zero and so there is no coupling between \( \dot{\varepsilon}_\theta \) and \( \dot{\sigma}_r \). Equations (4.29) and (4.30) therefore become:-

\[ \dot{\varepsilon}_r = \frac{\dot{\sigma}_r}{E} \]  

(4.32)

\[ \dot{\varepsilon}_\theta = 0 \]  

(4.33)

Thus the material stiffness matrix is:-

\[
[D] = \begin{bmatrix}
E & 0 \\
0 & 0
\end{bmatrix}
\]  

(4.34)

**Case (3):** \( \sigma_r = 0 \) and \( \sigma_\theta > 0 \)

The membrane is buckled in the tangential direction and because there is no coupling between \( \dot{\varepsilon}_r \) and \( \dot{\sigma}_\theta \) equations (4.29) and (4.30) become:-

\[ \dot{\varepsilon}_r = 0 \]  

(4.35)

\[ \dot{\varepsilon}_\theta = \frac{\dot{\sigma}_\theta}{E} \]  

(4.36)
Thus the material stiffness matrix is:

\[
[D] = \begin{bmatrix}
0 & 0 \\
0 & E \\
\end{bmatrix}
\]  \hspace{1cm} (4.37)

**Case (4):** \( \sigma_t = 0 \) and \( \sigma_\theta = 0 \)

The membrane is buckled in both the tangential and circumferential directions and therefore the material stiffness matrix becomes:

\[
[D] = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]  \hspace{1cm} (4.38)

### 4.2.4 Stress Update Calculations

At the beginning of each load step of the analysis the global incremental element stiffness matrix \([K_G]\) is calculated and the stiffness equations are inverted to give the global incremental nodal velocities. The Gauss point tangential and circumferential strain increments are then calculated by integrating equations (4.13) and (4.14) exactly, over the finite time increment \(\Delta t\) (corresponding to the load increment), to give:

\[
\Delta \varepsilon_t = \log_e \left[ \frac{J_{t+\Delta t}}{J_t} \right]
\]  \hspace{1cm} (4.39)

and:

\[
\Delta \varepsilon_\theta = \log_e \left[ \frac{r_{t+\Delta t}}{r_t} \right]
\]  \hspace{1cm} (4.40)

where \(\Delta \varepsilon_t\) and \(\Delta \varepsilon_\theta\) are the respective finite tangential and circumferential Hencky strain increments at a point, over the time increment \(\Delta t\). The value of the Jacobian at the beginning of the load step is \(J_t\) and at the end is \(J_{t+\Delta t}\), while the radius of the point at the beginning of the load step is \(r_t\) and at the end is \(r_{t+\Delta t}\).
For each increment of load the membrane thickness is determined by the expression:

\[
\log\left(\frac{T}{t_0}\right) = -\frac{\nu}{1-\nu} (\varepsilon_i + \varepsilon_0)
\]  (4.41)

where \( T \) is the thickness at the end of the load step, \( t_0 \) is the original reinforcement thickness and \( \varepsilon_i \) and \( \varepsilon_0 \) are the updated tangential and circumferential Hencky strains respectively.

At the end of a load step the updated Gauss point stresses associated with the calculated total strains are determined so that a check on the general equilibrium of the system can be undertaken. The total stresses are not path dependent, which means that they may be calculated directly from the total strains using equations of the same form as the stress-strain rate equations in Section 4.2.3.

Firstly it is assumed that case (1) applies, i.e. that both the updated tangential and circumferential Hencky stresses, \( \sigma_i \) and \( \sigma_0 \), are positive. Therefore equation (4.23) is used with \([D]\) given by equation (4.31):

\[
\sigma_i = \frac{E}{1-\nu^2} (\varepsilon_i + \nu \varepsilon_0)
\]  (4.42)

\[
\sigma_0 = \frac{E}{1-\nu^2} (\varepsilon_0 + \nu \varepsilon_i)
\]  (4.43)

where \( \varepsilon_i \) and \( \varepsilon_0 \) are the updated tangential and circumferential Hencky strains calculated using equations (4.39) and (4.40).

However, if either or both of the Gauss point stresses are found to be negative, then that stress is set to zero and they are both re-calculated for the appropriate case (2, 3 or 4) using the relevant material stiffness matrix, as described in Section 4.2.3.

4.3 Plane Strain Membrane Element Formulation

The large strain finite element formulation of the one-dimensional plane strain isoparametric membrane element, already utilised in OXFEM, is fully described in both Burd (1986) and
Burd and Houlsby (1986). The finite element equations derived for this plane strain element will therefore be discussed only briefly in this section to highlight the essential differences between the plane strain and axisymmetric formulations.

In the plane strain analysis the strain in the direction perpendicular to the $x$-$y$ plane is, by definition, zero. As in the axisymmetric case, the plane strain element is assumed to have no bending stiffness and therefore no shear stress component needs to be considered. The membrane is also assumed to be of no significant thickness, consequently the stresses normal to the membrane are negligible in comparison to the in-plane stresses. As a result, only the one component of longitudinal strain, $\varepsilon_L$, needs to be considered in the plane strain formulation.

The finite longitudinal Hencky strain increment, $\Delta \varepsilon_L$, at any particular point in the element, is found by a similar procedure to that described for the tangential Hencky strain increment within the axisymmetric membrane formulation, (see the derivation of equation (4.39)).

In calculating the plane strain nodal force rate vector $\mathbf{\tilde{P}}$, the incremental stiffness matrix $[K]$, can be algebraically manipulated into a convenient expression which is cast in the familiar form of the small displacement finite element equations, i.e.:-

$$[K] = \int_E [B]^*^T [D]^* [B]^* \, dV \quad (4.44)$$

where $[D]^*$ is a modified material stiffness matrix:

$$[D]^* = \begin{bmatrix} E^* & 0 \\ 0 & \sigma_L \end{bmatrix} \quad (4.45)$$

which contains the variable:

$$E^* = \frac{E}{1-\nu^2} - \frac{\sigma_L \nu}{1-\nu} \quad (4.46)$$
such that $E$ is the Young's modulus of the membrane material, $v$ is the Poisson's ratio and $\sigma_L$ is the longitudinal stress.

Furthermore, the modified matrix $[B]^*$ in equation (4.44) is of the form:

$$
[B]^* = \frac{1}{J^2} \begin{bmatrix}
X^T [N']^T [N'] \\
X^T [N']^T [Z]^T [N']
\end{bmatrix}
$$

(4.47)

where:

$$
[Z] = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
$$

(4.48)

and $X$ is the vector of global nodal co-ordinates, $[N]$ is the shape function matrix and $J$ is the Jacobian of the transformation from global to reference co-ordinates.

The plane strain incremental stiffness formulation, equation (4.44), is clearly of a much simpler configuration than that of the axisymmetric formulation, equation (4.22). This has the advantage of making the plane strain finite element implementation relatively easy in comparison to the more complicated axisymmetric formulation. Moreover, having $[K]$ in the small displacement form means that this large displacement plane strain formulation can be incorporated into existing finite element codes with the minimum of software modification.

An important feature of the plane strain incremental stiffness matrix is the (2 x 1) configuration of the modified matrix $[B]^*$. This matrix relates the nodal velocity vector $\dot{\mathbf{U}}$ to the strain rate vector $\dot{\epsilon}$, as shown in the axisymmetric equation (4.15). When considering the axisymmetric problem there are clearly two terms within the strain rate vector, those of the tangential and circumferential strain rates, $\dot{\epsilon}_r$ and $\dot{\epsilon}_\theta$, whereas for plane strain the two terms of the strain rate vector are not, initially, as obvious. The first term is
in fact the longitudinal strain rate \( \dot{\varepsilon}_L \), while the second term is actually identified with the anti-clockwise rotation rate of the element. Therefore, the plane strain nodal force rate vector, \( \dot{P} \), can be associated with both longitudinal straining and element rotation.

It is also useful to consider the significance of the stress rate vector \( \dot{\sigma} \), which is given by the product of the plane strain modified material stiffness matrix \( [D]^* \) and the strain rate vector \( \dot{\varepsilon} \), as shown in the axisymmetric equation (4.23). Clearly there are also two terms within the plane strain stress rate vector. The first term is the longitudinal stress rate \( \dot{\sigma}_L \), while the second stress rate term is associated with the element rotation rate.

Within the plane strain formulation an allowance is made for changes in the cross-sectional area of the membrane, \( A \), as the analysis progresses, similar to the axisymmetric formulation where account is made of the varying membrane thickness. It is necessary to calculate the cross-sectional area of the membrane, at the relevant Gauss point, in order to fully evaluate the plane strain incremental stiffness matrix \( [K] \), equation (4.44). For each increment of load the cross-sectional area is determined by the expression:-

\[
\log_e \left[ \frac{A}{A_0} \right] = -\frac{\nu}{1-\nu} \Delta \varepsilon_L \tag{4.49}
\]

where \( A_0 \) is the area at the start of the calculation step, \( A \) is the area at the end of the step and \( \Delta \varepsilon_L \) is the longitudinal Hencky strain increment.

All of the large displacement reinforced plane strain computations described in this thesis were carried out using a three-noded membrane element, with a three-point Gaussian integration rule. It is worthy of note that a two-point Gaussian integration rule would in fact be sufficient for the small displacement case, provided that the Jacobian did not vary along the element. The finite element equations for the specific case of the three-noded element are fully detailed in Burd (1986) and therefore require no repetition here.
4.4 Verification of the Axisymmetric Membrane Formulation

To validate the new axisymmetric membrane formulation and the corresponding computer code, several different test analyses are undertaken. The principal objective in conducting these relatively simple, but rigorous, tests is to verify that the membrane is modelled accurately before attempting the more complex problem of the reinforced two-layer soil system.

The test problems are designed to investigate the small and large displacement behaviour of the elastic membrane, since both these modes of deformation are fundamentally important within a reinforced two-layer soil system, for displacement and force controlled calculations. The membrane elements are also arranged in different configurations.

4.4.1 Small Strain Thick-walled Cylinder Analysis

To ensure that the finite element formulation predicts the tangential and circumferential stresses accurately, a relatively simple small strain elastic test is conducted. This problem is analogous to a small strain thick-walled cylinder of inner radius \(a = 1 \, \text{m}\) and outer radius \(b = 5 \, \text{m}\), subjected to an external tensile stress \(P^*\). The mesh used for the calculation consists of two five-noded axisymmetric membrane elements, as shown in Figure 4.4.

The material is allocated a relatively high value of Young’s modulus, \(E = 10^6 \, \text{kPa}\), to ensure a small displacement response, the Poisson’s ratio is taken as zero and the membrane thickness is set to unity. The expansion of the cylinder is carried out as a displacement controlled analysis.

The resulting tangential and circumferential stresses at each of the Gauss point co-ordinates \(r\) are plotted in Figure 4.4, normalised by the external stress \(P^*\). The well known elasticity solutions (e.g. Benham and Crawford (1987)) are also shown for comparison. It is found that the finite element results closely approximate the exact solution for the small strain elastic case.
Figure 4.4: Stresses in a Thick-walled Cylinder

4.4.2 Large Strain Cavity Expansion Analysis

This test problem is essentially a large displacement one-dimensional cavity expansion analysis. An exact closed form solution for this problem has been established by Sagaseta (1984), based on elastic perfectly-plastic large strain theory. The axisymmetric membrane element, however, has been defined as an elastic material and consequently this analytical solution needs to be modified to negate the plasticity part of the calculation.

Clearly a cavity expansion analysis requires the material to sustain compressive stresses, which the membrane material is specifically not allowed to do. However, the stress update calculations as described in Sections 4.2.3 and 4.2.4 can be modified to allow compression, so that the induced negative strains and stresses are not set to zero and where case (1) only of the constitutive behaviour applies. The value in undertaking this analysis is that the large displacement nature of the membrane element can be compared to a known solution. In all other uses of the axisymmetric membrane element in this thesis the original constitutive behaviour and stress updating apply.

The Sagaseta (1984) solution relates specifically to an incompressible material, i.e. one with a Poisson’s ratio of 0.5, but to avoid numerical instability the finite element
calculations must use the approximation of \( v = 0.49 \) for the membrane material. The analytical theory also assumes the continuum is of infinite extent. However, Burd and Houlsby (1990) show how this can in fact be simulated in a finite element analysis through using a "correcting layer" at the periphery of the mesh. By incorporating an additional element layer, with appropriately different material properties, it is possible to model an infinite medium while using a finite mesh size. This is achieved by adding a layer of elements, with a rigidly fixed edge of outer radius \( b \) to the mesh of initial outer radius \( c \), provided that the following relationship is satisfied:

\[
\frac{G^*}{G} = \frac{\left( \left( \frac{b}{c} \right)^2 - 1 \right) (1 - 2v^*)}{1 + \left( \frac{b}{c} \right)^2 (1 - 2v^*)}
\]  

(4.50)

where \( G \) is the shear modulus of the medium to be modelled and \( G^* \) and \( v^* \) are the shear modulus and Poisson's ratio of the correcting layer.

The one-dimensional mesh, of initial thickness \( t_0 = 0.0001 \), used for the finite element calculation is shown in Figure 4.5. The elastic membrane is assumed to have a shear modulus of 100 kPa and the correcting layer, of radius twice that of the original mesh radius, is prescribed the properties of \( G^* = 50 \) kPa and \( v^* = 0.25 \). The initial cavity radius \( a_0 \) is expanded to a final cavity radius of \( a = 5a_0 \), using a displacement controlled analysis.

![Figure 4.5: Finite Element Mesh for Cavity Expansion](image)

Figure 4.5: Finite Element Mesh for Cavity Expansion
Figure 4.6 shows the tangential and circumferential stresses calculated by the finite element analysis, compared to the Sagaseta (1984) analytical solution. The stresses are normalised by the shear modulus $G$, $a$ is the final cavity radius and $r$ is the radius corresponding to the position of the Gauss point stresses.

![Stress vs. $\frac{a}{r}$ graph]

**Figure 4.6: Stresses for Cavity Expansion Analysis**

The finite element predictions using 100 load steps generally agree well with the analytical solution. Some deviation of the results occurs at the inner boundary ($a/r = 1$) due to discretization errors. It was found that the results are very sensitive to the number of load steps used, causing increased divergence of the stresses from the Sagaseta (1984) solution for too many or too few steps.

### 4.4.3 Thin-walled Cylinder Expansion Analysis

A 2 m tall thin-walled cylinder of inner radius $R = 1$ m is modelled by two vertical axisymmetric membrane elements as shown in Figure 4.7. The cylinder wall, of initial thickness $t_0 = 0.001$ m, is assigned a Young's modulus of $E = 2 \times 10^8$ kPa and a Poisson's ratio $\nu = 0.3$.  

4-21
Keeping the tangential strain zero and neglecting any radial stress in the cylinder wall, it can be shown that for a uniform radial wall displacement of \( u_r \), the circumferential, \( \sigma_\theta \), and meridional, \( \sigma_r \), stresses in the cylinder are given by:

\[
\sigma_\theta = \frac{E}{(1 - \nu^2)} \ln \left( \frac{R + u_r}{R} \right)
\]

(4.51)

\[
\sigma_r = \nu \sigma_\theta
\]

(4.52)

where the meridional stress is identical to the tangential stress in the membrane.

The computed large displacement finite element membrane stresses and the corresponding exact elastic solutions, shown in Figure 4.7 for different radial wall displacements, are very comparable.

\[ \text{Figure 4.7: Expansion of a Thin-walled Cylinder} \]

4.4.4 Annular Plate with Fixed Periphery and Inward Pressure

A horizontal simply-supported annular plate of initial thickness \( t_0 = 0.0005 \, m \), rigidly fixed at the outer radius of \( b = 5 \, m \) with an inward acting pressure of \( P \) at the inner radius \( a = 1 \, m \), is used to study the case when one stress falls to zero.
The plate is assigned a Young's modulus of \( E = 2 \times 10^8 \) kPa and a Poisson's ratio of \( \nu = 0.3 \). For a linear elastic material the tangential, or in this case the radial stress, \( \sigma_r \), will be tensile throughout the plate, but the circumferential stress, \( \sigma_\theta \), will be tensile only within a small area at the outside edge and tends to be compressive towards the centre. The membrane material does not allow compressive stresses, so the elastic solution to this problem is found by considering two separate zones in the plate and maintaining compatibility of the stresses at the boundary.

For the unbuckled outer zone the equations of equilibrium, compatibility of strain and displacements, and the stress-strain relationships of elasticity combine to give:

\[
\begin{align*}
\sigma_r &= A - \frac{B}{r^2} \\
\sigma_\theta &= A + \frac{B}{r^2}
\end{align*}
\]  
(4.53)

where \( A \) and \( B \) are constants. Whereas, for the circumferentially buckled inner zone, the stresses are calculated by equilibrium only and are of the form:

\[
\begin{align*}
\sigma_r &= \frac{D}{r} \\
\sigma_\theta &= 0
\end{align*}
\]  
(4.54)

where \( D \) is a constant.

The boundary conditions of the problem, where \( c \) is the radius of the limit of the circumferential buckling as shown in Figure 4.8, are: at \( r = a \), \( \sigma_r = P \); at \( r = c \), \( \sigma_\theta = 0 \) and \( \sigma_r \) is continuous; and at \( r = b \), \( \epsilon_\theta = 0 \). By applying these boundary conditions and maintaining compatibility of the stresses at \( r = c \), it may be shown that:

\[
c = \frac{b \left( 1 - \nu \right)^{\frac{1}{2}}}{\left( 1 + \nu \right)^{\frac{1}{2}}}
\]  
(4.55)

and that in the unbuckled outer zone the tangential and circumferential stresses are:

\[
\sigma_r = \frac{Pa}{2b \left( 1 - \nu^2 \right)^{\frac{1}{2}}} \left[ (1 + \nu) + (1 - \nu) \left( \frac{b}{r} \right)^2 \right]
\]  
(4.56)
\[ \sigma_0 = \frac{P a}{2 b (1 - v^2)^{\frac{3}{2}}} \left\{ (1 + v) - (1 - v) \left( \frac{b}{r} \right)^2 \right\} \] (4.57)

and in the buckled inner zone:

\[ \sigma_i = \frac{P a}{r} \quad \sigma_0 = 0 \] (4.58)

---

**Figure 4.8: Annular Plate with Fixed Periphery and Internal Inward Pressure**

The close correlation between the large strain finite element results, using a mesh of 8 axisymmetric membrane elements, and the elasticity solution is illustrated in Figure 4.8.

4.4.5 Annular Plate with Fixed Periphery and Vertical Displacements

To check that the incremental stiffness matrix, \([K]\), is correctly formulated in the finite element equations, this test problem compares the nodal forces computed by two separate methods.

A single axisymmetric membrane element is used to model an annular plate of internal radius \(a = 1.0 \text{ m}\) and external radius \(b = 5a\), with the outer edge fixed. The plate is assigned
the properties of a Young's modulus $E = 10^3$ kPa, Poisson's ratio of zero and an initial thickness of unity. Vertical displacements, $\delta_v$, are prescribed to each of the inner four nodes leaving the fifth node fixed, so that the element slopes down towards the centre-line, but remains straight with equal distances between the nodes in the tangential direction, as shown in Figure 4.9. For a particular set of nodal displacements, $U$, the resulting vector of nodal forces, $P$, in equilibrium with the Gauss point stresses, $\sigma$, is calculated exactly by the equation (4.59):

$$ P = \int_E [B]^T \sigma \, dV $$

(4.59)

which is derived from equation (4.17) and where the $[B]$ matrix is defined in Appendix 4A.

These nodal forces can also be calculated incrementally by the equation (4.60):

$$ \Delta P = [K] \Delta U $$

(4.60)

where $[K]$ is determined by equation (4.22) and the final horizontal and vertical nodal forces, $F_x$ and $F_y$, are given by summing the incremental nodal forces over a finite number of steps. By increasing the number of steps the results of equation (4.60) approximate more closely to those of equation (4.59).

A comparison of the computed nodal forces, using 10,000 steps for equation (4.60), is shown in Figure 4.9. There is an average error of less than 0.5%, as opposed to $\approx 3\%$ for 1,000 steps.
Figure 4.9: Nodal Forces for varying Vertical Nodal Displacements

4.4.6 Expanding Sphere Analysis

By modelling the displacement controlled expansion of a thin-walled sphere, of initial radius $R = 1.0 \text{ m}$ and thickness $t_0 = 0.001 \text{ m}$, the membrane elements are tested for a variety of geometric configurations.
Six axisymmetric membrane elements are orientated in a semi-circle about a centre-line, as shown in Figure 4.10, with horizontal fixities at the top and bottom. Each node is prescribed the same outward radial displacement, \( u_r \), corresponding to a uniform inflating pressure, so that the resulting meridional (tangential) and circumferential membrane stresses, \( \sigma_t \) and \( \sigma_\theta \), are identical.

By elasticity theory it may be shown that these stresses are given by:

\[
\sigma_t = \sigma_\theta = \frac{E}{(1 - v)} \ln \left( \frac{R + u_r}{R} \right)
\]  

(4.61)

where \( E = 2 \times 10^9 \) kPa and \( v \) are the Young's modulus and Poisson's ratio of the material, respectively.

The membrane stresses at different radial displacements, predicted by the large displacement finite element analysis, are compared in Figure 4.10 to the elasticity solutions for two different values of Poisson's ratio. The analyses were conducted using displacement control, since force control proved too unstable.

![Figure 4.10: Membrane Stresses in Expanding Sphere](image-url)
APPENDIX 4A

Finite Element Equations for the Five-Noded Membrane Element

Figure 4.11 shows the isoparametric mapping between the parent and reference elements for the five-noded membrane element.

Parent Element

Mapped Element

Figure 4.11: Mapping for the Five-Noded Axisymmetric Membrane Element

The vectors $r$ and $R$, that are related by equation (4.4), are defined as:

\[
\begin{align*}
\mathbf{r}^T &= \begin{bmatrix} r \\ z \end{bmatrix} \\
\mathbf{R}^T &= \begin{bmatrix} R_1 & Z_1 & R_2 & Z_2 & \ldots & R_5 & Z_5 \end{bmatrix}
\end{align*}
\]  

(4.62)  

(4.63)

Where $R_i, Z_i$ are the global co-ordinates of the $i^{th}$ node. The shape function matrix is:

\[
[N] = \begin{bmatrix} f_1 & 0 & f_2 & 0 & f_3 & 0 & f_4 & 0 & f_5 & 0 \\ 0 & f_1 & 0 & f_2 & 0 & f_3 & 0 & f_4 & 0 & f_5 \end{bmatrix}
\]  

(4.64)

in which:

\[
f_1 = \frac{2}{3} (\alpha + 1/2) (\alpha) (\alpha - 1/2) (\alpha - 1)
\]

(4.65)

\[
f_2 = \frac{2}{3} (\alpha + 1/2) (\alpha) (\alpha - 1/2) (\alpha + 1)
\]

(4.66)
\[ f_3 = -\frac{8}{3} (\alpha + 1) (\alpha) (\alpha - 1/2) (\alpha - 1) \]  
(4.67)

\[ f_4 = 4 (\alpha + 1) (\alpha - 1) (\alpha + 1/2) (\alpha - 1/2) \]  
(4.68)

\[ f_5 = -\frac{8}{3} (\alpha + 1) (\alpha) (\alpha + 1/2) (\alpha - 1) \]  
(4.69)

The \( [B] \) matrix of equation (4.15) is given by:

\[ [B] = \begin{bmatrix}
\frac{f_1'}{r^2} & \frac{f_2'}{r^2} & \frac{f_2'}{r^2} & \frac{f_3'}{r^2} & \cdots & \frac{f_5'}{r^2} & \frac{f_5'}{r^2} \\
0 & \frac{f_2}{r} & 0 & \cdots & 0 & \frac{f_5}{r}
\end{bmatrix} \]  
(4.70)

The \( [C_1] \) matrix of equation (4.26) maybe shown to be:

\[ [C_1] = \begin{bmatrix}
C_{1,1} & C_{1,2} & \cdots & C_{1,10} \\
C_{2,1} & C_{2,2} & \cdots & C_{2,10} \\
C_{10,1} & C_{10,2} & \cdots & C_{10,10}
\end{bmatrix} \]  
(4.71)

in which:

\[ C_{2i-1,2j-1} = \sigma, f_i' f_j' A \frac{j^2}{j^2} - \frac{\sigma_0 f_i f_j}{r^2} \]  
(4.72)

\[ C_{2i,2j-1} = 2 \sigma, f_i' f_j' B \frac{j^2}{j^2} \]  
(4.73)

\[ C_{2i-1,2j} = 2 \sigma, f_i' f_j' B \frac{j^2}{j^2} \]  
(4.74)

\[ C_{2i,2j} = -\sigma, f_i' f_j' A \frac{j^2}{j^2} \]  
(4.75)

where

\[ A = \frac{1}{j^2} \left( \left( \frac{dz}{d\alpha} \right)^2 - \left( \frac{dr}{d\alpha} \right)^2 \right) \]  
(4.76)

and

\[ B = -\frac{1}{j^2} \left( \left( \frac{dz}{d\alpha} \right) \left( \frac{dr}{d\alpha} \right) \right) \]  
(4.77)
CHAPTER 5:

ANALYSIS OF THE SOIL-REINFORCEMENT INTERFACE

5.1 Introduction

This chapter is concerned with the analysis of the soil-reinforcement interface for a two-layer soil system, using both standard continuum elements and a special class of finite element termed interface elements. In Section 5.2 a description is given of the methods that might be used to extract the interface stresses from a finite element solution for the case when no interface elements are used. A brief review of different types of interface elements currently available is given in Section 5.3 and the formulation of a new large strain, small relative displacement, axisymmetric interface element is described in Section 5.4.

To investigate fully the mechanisms of reinforcement within a two-layer soil system, which incorporates a geosynthetic membrane, the stresses acting at the interfaces of the different materials need to be calculated accurately. This requires an exact modelling of the soil-structure interaction.

In previous publications on finite element studies for reinforced two-layer soil systems using OXFEM, it was assumed that the reinforcement interlocked perfectly with the surrounding soil and that no slip occurred at the soil-reinforcement interfaces (e.g. Burd (1986), Burd and Houltsby (1989) and Burd and Brocklehurst (1990)). However, this assumption was shown by the experimental work of Fannin (1986) to be justified only for the area directly under the footing (see Section 2.3). Outside of this region it was found that the interlock between the granular material and the grid reinforcement was partially lost due
to the upward heave movements of the deforming subgrade through the grid. It was also reported by Milligan et al. (1986) that when excavations were conducted of the full-scale plate loading tests performed on reinforced two-layer soil systems, little evidence was actually found of the geogrid interlocking with particles of fill. Therefore, to model numerically the soil-reinforcement interfaces more accurately it was felt necessary to include interface elements in OXFEM.

The use of interface elements has two distinct advantages. Firstly they allow the properties of the reinforcement-soil interface to be varied independently of the soil properties and secondly they provide a convenient way of extracting the stresses acting at the soil-reinforcement interface. All of the plane strain and axisymmetric finite element analyses, of two-layer soil systems, discussed in later chapters of this thesis have interface elements incorporated.

5.2 Determination of Interface Stresses from Continuum Elements

The key to understanding the mechanisms of reinforcement of a two-layer soil system is a clear knowledge of the shear stresses acting at the interface. It is important, therefore, to determine these stresses precisely.

Difficulties in accurately calculating and analysing the interface stresses at the continuum-membrane element boundaries were discussed in the finite element analyses reported in Burd (1986). The method that was used by Burd (1986) to determine these interface stresses, from the six-noded continuum elements and three-noded membrane elements, involved calculating them from the force associated with the mid-side node of the common edge. The forces at the element corners were ignored, as discussed later for Method 2. It was found that the shear stresses acting at the membrane surface, for both reinforced and unreinforced analyses, showed marked variations along the length of the interface, whereas the normal stresses were much better conditioned. These large spatial variations of shear stress were attributed by Burd (1986) to discretization errors involved in the numerical model and to the general tendency of the finite element method to produce
solutions which have local erroneous variations even though they are globally correct. It was felt, however, that an improved method of interface analysis was needed, especially since the shear stresses are of particular importance.

An investigation, as part of this dissertation, into the different ways of establishing the normal, $\sigma$, and shear, $\tau$, stresses at an interface between continuum and membrane elements has led to the development of four separate methods of analysis. Figure 5.1 depicts a typical boundary between a six-noded, plane strain, continuum element containing three Gauss points ($A$, $B$ and $C$) and a three-noded membrane element of length $L^m$. The continuum-membrane element boundary is illustrated here for the case of small displacement (i.e. the nodal positions are not updated), with the reinforcement horizontal.

![Diagram of continuum-membrane element boundary](image)

**Figure 5.1: A Typical Continuum-Membrane Element Boundary**

*(Without an Interface Element)*

**Method 1:** The interface stresses are calculated from the contribution of the forces acting at all three of the nodes along the side of each six-noded continuum element directly adjacent to the membrane. The method assumes that 100% of the central nodal force and that 50% of the edge nodal forces contribute to the overall stress for that element. For the two continuum elements at the membrane ends, the full 100% of the edge nodal forces are taken into consideration (rather than 50%). Thus, the general boundary stresses are:-
\[ \sigma = \frac{(F_{1y} + F_{3y})/2 + F_{2y}}{L^m} \]  
(5.1)

\[ \tau = \frac{(F_{1x} + F_{3x})/2 + F_{2x}}{L^m} \]  
(5.2)

except for the left edge continuum-membrane elements, where the stresses are given by:-

\[ \sigma = \frac{F_{1y} + (F_{3y})/2 + F_{2y}}{L^m} \]  
(5.3)

\[ \tau = \frac{F_{1x} + (F_{3x})/2 + F_{2x}}{L^m} \]  
(5.4)

where \( F_{ix} \) and \( F_{iy} \) are the nodal forces at node \( i \) in the \( x \) and \( y \) directions respectively.

**Method 2:** Considers only the force at the mid-side node of the same continuum elements that are in contact with the membrane. It uses the equivalent nodal force loading ratios specific to a three-noded edge element, where a ratio of 2/3 of the total force applied to the element side acts at the mid-side node and 1/6 acts at each end node, therefore:-

\[ \sigma = \frac{3F_{2y}}{2L^m} \]  
(5.5)

\[ \tau = \frac{3F_{2x}}{2L^m} \]  
(5.6)

**Method 3:** This method considers a mean value for the three Gauss point stresses of the boundary continuum elements. Care must be taken for large strain, large displacement analyses to ensure that the normal and shear stresses obtained are actually perpendicular and parallel to the deformed membrane respectively, due to element rotation:-

\[ \sigma = \{ (\sigma_y)_A + (\sigma_y)_B + (\sigma_y)_C \} / 3 \]  
(5.7)

\[ \tau = \{ (\tau_{xy})_A + (\tau_{xy})_B + (\tau_{xy})_C \} / 3 \]  
(5.8)
where \((\sigma_y)_A\) is the value of the normal stress at Gauss point \(A\) and \((\tau_{xy})_A\) is the value of the shear stress at Gauss point \(A\).

**Method 4:** This is similar to Method 3, but uses only the two Gauss points nearest the reinforcement in the continuum element concerned and it again requires the relevant stresses to be modified due to large strain element rotation:

\[
\sigma = \frac{1}{2} \left\{ (\sigma_y)_B + (\sigma_y)_C \right\} \tag{5.9}
\]

\[
\tau = \frac{1}{2} \left\{ (\tau_{xy})_B + (\tau_{xy})_C \right\} \tag{5.10}
\]

For all of the methods discussed above the position at which the calculated stress acts is taken to be the average of the horizontal co-ordinates of the relevant nodes, or Gauss points considered. Typical graphs of the various normal and shear stresses, calculated at the upper and lower interfaces between continuum and membrane elements, are presented in Figures 5.3 and 5.4. These interface stresses are extracted from one small strain finite element calculation, using each of the four methods of analysis. In addition, the stresses obtained directly from an identical finite element calculation using interface elements are also illustrated. The stresses are normalised with respect to the plane strain undrained shear strength of the clay, \(s_u^{es}\), and the horizontal distance from the footing centre-line, \(x\), is normalised with respect to the footing half-width, \(B\).

The plane strain finite element mesh used for this soil-reinforcement interface study, shown in Figure 5.2, was developed by the mesh generator program OX-MESH (Houlby (1988)) and has overall dimensions of 5.0 \(m\) wide by 5.3 \(m\) deep with the reinforcement layer, separating the fill above from the clay below, at a depth of 0.3 \(m\) (shown as a double horizontal line). The mesh possesses 449 six-noded continuum elements and 20 three-noded membrane elements. The footing half width is \(B = 0.25\ m\) and the material properties of the clay, fill and reinforcement are as given in Appendix 6A for the central parametric analysis. The interface properties for the one comparative analysis are also listed in Appendix 6A.
The results of the study to investigate the four methods of interface analysis show that the normal stress profiles along the reinforcement length are all of a relatively comparable nature, Figure 5.3. Method 3, however, gave the most inconsistent result, in that the upper and lower interface normal stresses are not equal, whereas each of the other methods produced similar, reasonably smooth curves. The shear stress profiles, on the other hand, differed dramatically, Figure 5.4. Those obtained by using Methods 1 and 2 exhibited considerable oscillations and variations, especially for the lower shear stresses (which are thought to be spurious), whilst those from Methods 3 and 4 displayed a far smoother distribution. Method 1 gave the most disparate set of shear stresses of the four methods.
Interface Elements

Figure 5.3: Normal Stress Profiles along Interface for each Method of Analysis
Figure 5.4: Shear Stress Profiles along Interface for each Method of Analysis
It was concluded that the calculated normal stresses are approximately independent of the method, but that for the shear stresses the method based on the use of Gauss point stresses in adjacent elements gave results that were considered more satisfactory than those based on the use of nodal forces. Method 4 is thought to be the most appropriate of the four continuum element methods, since the calculated mean stresses in Method 3 will be slightly removed from the membrane, whereas in Method 4 the mean stresses are derived from the two Gauss points nearest the membrane. It is expected, however, that if a dense 'thin' band of continuum elements were placed above and below the reinforcement, across the entire width, then Method 3 may well be a more accurate analysis to use.

This study suggests that it is possible to extract interface stresses from the continuum elements. It was decided, however, to implement interface elements for each of the parametric study analyses (Chapters 6 - 10), since these provide a more straightforward approach to the extraction of interface stresses, this conclusion was also reached by Zeevaert (1980). A mean value of the Gauss point stresses can be obtained directly without concern for the large strain element rotations. Interestingly, the computed interface stresses at the end of a typical analysis using interface elements were found to be similar to those obtained by the best of the previous methods, i.e. Method 4, (see Figures 5.3 and 5.4). However, the use of interface elements is preferred on the basis that the stresses calculated are those precisely on the line of the interface, rather than those at a small distance from the reinforcement as generated by Method 4.

An investigative study into the modelling of interfaces using conventional eight-noded quadrilateral continuum elements was undertaken by Griffiths (1985). It was shown that satisfactory results for modelling both frictional and adhesive slippage at interfaces could be obtained, provided the thickness to length ratio of the elements remained smaller than 100. Slippage in the elements manifested itself by an increase in both the calculated shear displacements and the number of iterations necessary for numerical convergence. However, Griffiths (1985) further concluded that continuum elements at interfaces made good 'first approximations' in an analysis, but if large relative movement was a major factor, then specialised interface elements may prove necessary.
5.3 Types of Interface Elements

Conventional continuum elements have often proven inadequate in modelling soil-structure interaction exactly, especially where large relative lateral movement, or indeed separation occurs, e.g. Bell (1991) and Gens et al. (1989). However, the use of interface elements has been relatively successful in modelling such soil-structure interactions.

Interface elements have typically been applied to the problem of modelling joined rock masses (Goodman et al. (1968)), or to the simulation of particular soil-structure interface behaviour, such as footings (Bell (1991)), soil reinforcement (Handel et al. (1990)) and piles (Zaman (1985)). The first proposal of a specific interface element formulation was made by Goodman et al. (1968) for the purely elastic case. Subsequently four main categories of finite elements have been proposed for the modelling of joints and interfaces, as detailed by Gens et al. (1989):-

1) Standard solid finite elements of small thickness, i.e. with an aspect ratio of < 100 (Griffiths (1985)).

2) Quasi-continuum elements possessing a weakness plane in the direction of the interface plane (Zienkiewicz et al. (1970)).

3) Linkage elements in which only the connections between opposite nodes are considered (Frank et al. (1982)).

4) Interface elements in which relative displacements between opposite nodes are the primary deformation variables and which use elastic, or elasto-plastic constitutive laws. This class combines the ‘thin-layer element’ of finite thickness (Zaman (1985)) and the ‘zero-thickness element’ (Goodman et al. (1968), Gens et al. (1989) and Burd and Brocklehurst (1991)).

An alternative procedure for modelling a soil-structure interface is presented by Moore and Booker (1989) and Kodikara and Moore (1991) for the particular problem of buried flexible pipes. They suggest using a nonlinear frictional-adhesive zero-tension interface model at the boundary between the two separate bodies, rather than employing a discrete interface element. The technique permits large rotations and substantial slip between the
interacting bodies, as well as the occurrence, or loss, of contact at the interface. However, large relative slip is not an important characteristic of the two-layer soil system and therefore this interface model would be superfluous.

The zero-thickness interface element of category 4 has been used successfully within OXFEM for both the plane strain (Burd and Brocklehurst (1991)) and the three-dimensional (Bell (1991)) problem of modelling a distinct soil-structure interface. This type of element has therefore been incorporated into the finite element model described in Chapter 3. Details of the new axisymmetric interface element formulation, which is based on the same general procedures as those used for the plane strain interface element formulation described in Burd and Brocklehurst (1991), are presented in Section 5.4.

5.4 Axisymmetric Interface Element Formulation

The large strain finite element formulation of the new zero-thickness axisymmetric interface element is based on an isoparametric approach. The formulation takes account of the effects of large global displacements and rotations, but the relative displacements within the element are constrained to being infinitesimal. Although this is an apparent limitation to the general applicability of the element it is justified for the specific cases of the two-layer soil systems studied in this thesis, because in the important region immediately beneath the footing gross slip would not be expected to occur (as found experimentally by Fannin (1986) and numerically in Section 6.5).

The axisymmetric formulation described in this chapter is applicable to axisymmetric interface elements of arbitrary order and is therefore presented in a general form. All of the axisymmetric computations described in this thesis were conducted using a ten-noded axisymmetric interface element, since this is kinematically compatible with both the fifteen-noded axisymmetric continuum element and the five-noded axisymmetric membrane element previously described. The specific equations for the case of the ten-noded element are presented in Appendix 5A.
5.4.1 Kinematic Equations

A general axisymmetric interface element with N nodes is shown in Figure 5.5. The interface element geometry consists of pairs of nodes sharing the same co-ordinates, however, within each pair the nodes have independent degrees of freedom. The global cylindrical polar co-ordinates of a point in the element are \( r, z \) and these are mapped onto a single reference co-ordinate \( \alpha \), as shown in Figure 5.5.

![Diagram showing isoparametric mapping for an axisymmetric interface element](image)

**Figure 5.5: Isoparametric Mapping for an Axisymmetric Interface Element**

Figure 5.6(a) shows an infinitesimal portion of the interface element where two coincident arbitrary points, \( a \) and \( b \), lie on the element sides A and B. The vector of the global co-ordinates of the two points in the element is \( \bar{r} = [r_a \ z_a \ r_b \ z_b] \) and is related to the vector of global nodal co-ordinates \( R \) by:

\[
\bar{r} = [N] \ R
\]  

(5.11)

where \([N]\), the shape function matrix, is a function of \( \alpha \). (Note that the co-ordinates \((r_a, z_a)\) are identical to \((r_b, z_b)\).)
5.6(a) Global Velocities

5.6(b) Relative Velocities

Figure 5.6: Interface Element Velocities

In a similar manner the velocity vector of the same two arbitrary points, \( u^T = [u_a \ v_a \ u_b \ v_b] \), is related to the nodal velocity vector \( \underline{U} \) by:

\[
\underline{u} = [N] \underline{U}
\]

(5.12)

where \((u_a, v_a)\) and \((u_b, v_b)\) are the coincident velocities in the \(r\) and \(z\) directions at surfaces A and B respectively.

The velocities of the two congruent points on the interface may be resolved into directions parallel to, and perpendicular to, the local slope of the element, corresponding to the tangent and the normal of the element, Figure 5.6(b). The relative tangential \((u_r)\) and normal \((v_r)\) velocities within the interface are given by:

\[
\begin{align*}
    u_r &= (u_b - u_a) \cos \theta + (v_b - v_a) \sin \theta \\
    v_r &= (v_b - v_a) \cos \theta - (u_b - u_a) \sin \theta
\end{align*}
\]

(5.13) (5.14)
where \( \theta \) is the local angle of the interface relative to the \( r \) axis. The sign convention adopted here is that the relative tangential velocity is positive for the upper surface B moving to the right relative to the lower surface A and the normal velocity is positive when the interface surfaces move apart.

The vector of the relative velocities \( \mathbf{u}_r = [u_r, v_r] \) is given by the matrix equation:-

\[
\mathbf{u}_r = [S] [L] \mathbf{u}
\]  

(5.15)

where:-

\[
[L] = \begin{bmatrix}
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix}
\]  

(5.16)

and:-

\[
[S] = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
\]  

(5.17)

The components of the matrix \([S]\) can be rewritten in terms of the infinitesimal lengths of the element and differentiated with respect to \( \alpha \), giving:-

\[
\cos \theta = \frac{dr/d\alpha}{[ (dr/d\alpha)^2 + (dz/d\alpha)^2 ]^{1/2}}
\]  

(5.18)

and:-

\[
\sin \theta = \frac{dz/d\alpha}{[ (dr/d\alpha)^2 + (dz/d\alpha)^2 ]^{1/2}}
\]  

(5.19)

The denominator of both these expressions can be interpreted as \( J \), which represents the Jacobian of the transformation from parent co-ordinates \((r, z)\) to the reference co-ordinates \((\alpha)\). In this case equations (5.18) and (5.19) may be rewritten:-

\[
\cos \theta = \frac{1}{J} \frac{dr}{d\alpha}
\]  

(5.20)

and:-

\[
\sin \theta = \frac{1}{J} \frac{dz}{d\alpha}
\]  

(5.21)
Substituting these equations into matrix \([S]\) given in equation (5.17) gives:

\[
[S] = \frac{1}{J} \begin{bmatrix}
\frac{dr}{d\alpha} & \frac{dz}{d\alpha} \\
\frac{dz}{d\alpha} & \frac{dr}{d\alpha}
\end{bmatrix} = \frac{1}{J} \begin{bmatrix}
r' & z' \\
-z' & r'
\end{bmatrix}
\] (5.22)

where the prime denotes differentiation with respect to \(\alpha\).

Equation (5.15) can be rewritten in a more standard form by combining it with equation (5.12), giving:

\[
u_r = [B] \ U
\] (5.23)

where the matrix \([B]\) relates the vector of nodal velocities, \(U\), to the vector of relative velocities at a point in the interface, \(u_r\), and is given by:

\[
[B] = [S] \ [L] \ [N]
\] (5.24)

### 5.4.2 Constitutive Equations

The constitutive equations for this zero-thickness axisymmetric interface element are based on a Mohr-Coulomb elastic-perfectly frictional model of behaviour. Figure 5.7 schematically illustrates the Mohr-Coulomb yield function, \(f(\sigma)\), defined by:

\[
f = |\tau| + \sigma \tan \phi_i = 0
\] (5.25)

where \(\phi_i\) is the angle of friction of the interface.

There are only two co-ordinate directions considered in axisymmetry (i.e. \(r\) and \(z\)) and only two component stresses are acting, the shear stress, \(\tau\), and normal stress, \(\sigma\) as shown in Figure 5.7. There is no circumferential shear stress, because there is no relative velocity in that direction. Tensile normal stress is considered as positive.
Figure 5.7: Mohr-Coulomb Yield Surface

The vector of stress, $\sigma$, of an arbitrary point in the interface can be differentiated with respect to time (denoted by a superior dot) to give the stress rate vector, $\dot{\sigma} = [\dot{\tau} \quad \dot{\sigma}]$, which is then related to the relative velocities, $u_r$, by the expression:

$$\dot{\sigma} = [D]^{Ep} u_r$$

$$= [D]^{Ep} [B] U$$  \hspace{1cm} (5.26)

where $[D]^{Ep}$ is the elasto-plastic material stiffness matrix and is expressed in terms of the elastic component of the material stiffness matrix $[D]^E$ and the plastic component $[D]^P$ thus:

$$[D]^{Ep} = [D]^E + [D]^P$$  \hspace{1cm} (5.27)

Note that the interface element relative velocities are analogous to strain in continuum mechanics.

The elastic material stiffness matrix is essentially the same as that proposed in the
original work by Goodman et al. (1968), which is given by:-

\[
[D]^E = \begin{bmatrix} k_s & 0 \\ 0 & k_n \end{bmatrix}
\]  

(5.28)

The parameters \( k_s \) and \( k_n \) represent the shear and normal stiffnesses of the interface, respectively.

The plastic material interface stiffness matrix is derived from the approach conventionally adopted for continuum elements (e.g. Burd (1986)) and is given by:-

\[
[D]^P = -[D]^E \left( \frac{\delta \sigma}{\delta \tau} \right)^T \left( [D]^E \right)^{-1}
\]  

(5.29)

This equation (5.29) involves the assumptions that the relative velocities can be decomposed into elastic and plastic parts and that the stress rate is related to the elastic relative velocities. The Mohr-Coulomb yield function, \( f(\sigma) \), is given in equation (5.25) and \( g(\sigma) \) is the plastic potential which is used to define the plastic relative velocities using equations of the form:-

\[
u^p_r = \lambda \frac{\delta g}{\delta \tau}
\]  

(5.30)

\[
u^p_i = \lambda \frac{\delta g}{\delta \sigma}
\]  

(5.31)

where \( \lambda \) is an undetermined multiplier. The plastic potential developed in this analysis is assumed to be non-associated, where:-

\[
g = |\tau| + \sigma \tan \psi_i
\]  

(5.32)

and \( \psi_i \) is the angle of dilation of the interface. A non-associated flow rule allows the dilation characteristics of the material to be specified independently from the friction angle.

If the Gauss point stress lies inside the yield surface, then the plastic stiffness matrix is set to zero, but if the stress point lies on the yield surface then \([D]^P\) is:-

\[
[D]^P = \frac{k_s k_x}{k_s + k_n \tan \phi_i \tan \psi_i} \begin{bmatrix} \tan \phi_i \tan \psi_i & -\text{sgn}(\tau) \tan \phi_i \\ -\text{sgn}(\tau) \tan \psi_i & 1 \end{bmatrix}
\]  

(5.33)
where \( sgn(\tau) = 1 \) for \( \tau > 0 \) and \( sgn(\tau) = -1 \) for \( \tau < 0 \).

### 5.4.3 Stress Update Calculations

The finite element calculations are typically conducted as a series of steps for a particular prescribed displacement, or prescribed force, or indeed a mixture of both. At the end of each load step, corresponding to a finite time increment \( \Delta t \), the updated normal and shear stresses associated with the calculated relative velocities are determined by a 'stress update procedure'. This provides for a check on the general equilibrium of the system and also gives the out-of-balance nodal forces that are needed for the incremental solution procedure. The computed final stresses are characterised into different states according to their position in stress space relative to the Mohr-Coulomb failure envelope shown in Figure 5.8. The three possible states are denoted as Elastic, Plastic and Relaxed as indicated in Figure 5.8.

At the start of a load step \( \Delta t = 0 \) and at the end \( \Delta t = 1 \). In order to calculate the Gauss point stresses at the end of a load step the response of each Gauss point is assumed, initially, to be elastic, regardless of initial state. The relevant final stresses, \( \tau \) and \( \sigma \), are calculated using the elastic material stiffness matrix \( [D]^E \) in equation (5.26), giving:

\[
\begin{align*}
\tau &= \tau_0 + k_s \, u, \, \Delta t \\
\sigma &= \sigma_0 + k_n \, v, \, \Delta t
\end{align*}
\]  

(5.34)  

(5.35)

where \( \tau_0 \) and \( \sigma_0 \) are the initial Gauss point stresses and \( \Delta t = 1 \) at the end of the load step.

This assumption of a purely elastic stress update is then checked to ensure that the final stresses, \( \tau \) and \( \sigma \), have not violated the failure conditions:

1) If \( |\tau/\sigma| \geq \tan \phi \), then the updated stress point has yielded and must be re-analysed plastically beyond the point that the stress path intersects the Mohr-Coulomb yield surface.

2) If \( \sigma \geq 0 \) then separation of the interface has occurred during the load step and the stress point must be relaxed by setting the stresses to zero, i.e. \( \sigma = \tau = 0 \).
Figure 5.8: Mohr-Coulomb Yield Surface and Possible cases of Constitutive Behaviour

There are five possible cases of constitutive behaviour, realising that the initial stress state is either elastic and lies inside the failure envelope, or plastic and lies on the failure envelope, as illustrated in Figure 5.8.

Case (i)

The calculated stresses at the end of the assumed elastic step are **correct**.

Case (ii)

The Gauss point stress is initially on the failure envelope, but the final stresses are correct since the behaviour involves a purely elastic increment away from the yield surface.

Case (iii)

As in case (ii) the initial Gauss point stress lies on the failure envelope, but for this case the assumed elastic increment produces stresses outside of the yield surface. The entire step, therefore, has to be re-analysed plastically using the elasto-plastic material stiffness matrix \([D]^{EP}\) in equation (5.26), so that the final stresses are:
\[ \tau = \tau_0 + C (u, \tan \phi_i \tan \psi_i - v, \text{sgn}(\tau_0) \tan \phi_i) \Delta t \]  
\[ \sigma = \sigma_0 + C (-u, \text{sgn}(\tau_0) \tan \psi_i + v, \Delta t \]  

where the constant, \( C \), is:

\[ C = \frac{k_n k_s}{k_s + k_n \tan \phi_i \tan \psi_i} \]  

**Case (iv)**

The initial stress point is elastic, but the elastic updated stresses lie outside of the yield surface. The proportion of the time increment, \( \beta \), at which the stresses crossed the yield surface must be determined from the equation given by combining equations (5.34), (5.35) and the yield function equation (5.25):

\[ \frac{(\tau_0 + k_s u, \beta \Delta t)^2}{(\sigma_0 + k_n v, \beta \Delta t)^2} = \tan^2 \phi_i \]  

for which there will be only one positive real root and \( 0 < \beta < 1 \).

The elastic and plastic parts of the stress increment can then be calculated separately for the time increments \( \beta \Delta t \) and \( (1 - \beta) \Delta t \), respectively. The yield point stresses, \( \tau_y \) and \( \sigma_y \), are calculated from the elastic equations (5.40) and (5.41):

\[ \tau_y = \tau_0 + k_s u, \beta \Delta t \]  
\[ \sigma_y = \sigma_0 + k_n v, \beta \Delta t \]  

and the final plastic stresses, using \([D]^{ep}\), are:

\[ \tau = \tau_y + C (u, \tan \phi_i \tan \psi_i - v, \text{sgn}(\tau_y) \tan \phi_i) (1 - \beta) \Delta t \]  
\[ \sigma = \sigma_y + C (-u, \text{sgn}(\tau_y) \tan \psi_i + v, (1 - \beta) \Delta t \]  

where the constant, \( C \), is given in equation (5.38).
Case (v)

An additional consideration when correcting a yielded, or relaxed stress point back onto the yield surface is whether the plastic stresses have passed through the origin, i.e. the plastic normal stress becomes positive. In this situation the final plastic stresses must remain at zero, since the shear stress cannot change sign. This particular case of constitutive behaviour is depicted in Figure 5.8.

The elastic and plastic stress updating is initially done by the method described for case (iv), but if \( \sigma \geq 0 \) by equation (5.43) then both final stresses are set to zero, i.e. \( \sigma = \tau = 0 \).

If \( \gamma \) is the proportion of the time increment between the point of yield surface intersection and the origin, then the precise time at which the shear stress passes through the origin is given by:-

\[
(\beta + \gamma) \Delta t = \frac{\tau_y - \tau_0}{k_x u_r} - \frac{\tau_y}{C \left( \psi_i \tan \phi_i \tan \psi_i - \nu, \text{sgn}(\tau_y) \tan \phi_i \right)}
\]

(5.44)

where \( 0 < \gamma < 1 \), equation (5.40) gives \( \tau_y \) and \( C \) is given in equation (5.38).

These updated final stresses are used for the equilibrium check to find the new nodal forces in the interface element.

5.4.4 Finite Element Equations

If a set of virtual velocities \( \tilde{U}^* \) are applied to the nodes of the element, then by the principle of virtual work:-

\[
\tilde{U}^{*T} \tilde{P} = \int_\mathcal{E} \tilde{u}_r^{*T} \tilde{\sigma} \ dA
\]

(5.45)

where \( \tilde{u}_r^* \) are the relative velocities compatible with the virtual nodal velocities, \( \tilde{U}^* \), and \( \tilde{P} \) is the vector of nodal forces which are in equilibrium with the stresses \( \tilde{\sigma}^T = [ \tau \ \sigma ] \). The integration is performed over the surface area of the element.
The elemental surface area, \( dA \), is given by:

\[
dA = 2\pi r J \, d\alpha
\]  

(5.46)

It follows, therefore, that the virtual work equation (5.45) can be rewritten, substituting for the relative velocities vector as given in equation (5.23), as:

\[
U^* \sigma_0 \sigma J \, d\alpha = 2\pi \int_E U^T [B]^T \sigma r J \, d\alpha
\]  

(5.47)

Since this expression must hold for arbitrary nodal velocities \( U^* \), it follows that:

\[
P = 2\pi \int_E [B]^T \sigma r J \, d\alpha
\]  

(5.48)

where the scalar variable \( r \), the radius, is conveniently expressed by the matrix equation:

\[
r = k \, [N] \, R
\]  

(5.49)

where \( k \) is the constant vector \([1 \ 0 \ 1 \ 0]\).

Equation (5.48) is differentiated with respect to time to give the nodal force rate vector:

\[
\dot{P} = 2\pi \int_E \left\{ [B]^T \ddot{\sigma} r J + ([B]^T J) \sigma r + [B]^T \sigma J \right\} \, d\alpha
\]  

(5.50)

Note that in the second term of the integrand the superior dot relates to the product \( [B]^T J \).

In order to use this expression to derive an element stiffness matrix it is necessary to express each component of the integrand in terms of the vector of nodal velocities, \( U \), so that:

\[
\dot{P} = [K] \, U
\]  

(5.51)

where \([K]\) is the incremental stiffness matrix given by:
\[ [K] = 2 \pi \int_E \{ [B]^T [D] \, [D]^E \, [B] \, r \, J + [C] \} \, d\alpha \]  \hspace{1cm} (5.52)

The first term in the integrand of the incremental stiffness matrix is the conventional infinitesimal displacement stiffness matrix obtained by substituting equation (5.26) into equation (5.50), i.e.:

\[ [B]^T \, \dot{\sigma} \, r \, J = [B]^T [D] \, \dot{[B]} \, U \, r \, J \]  \hspace{1cm} (5.53)

The second component in equation (5.52) is the distortion rate matrix, \([C]\), which contains additional terms to account for the rate of change of element geometry, as proposed by Burd (1986). These need to be included in the stiffness matrix to model the large strain and large displacement effects. Equation (5.52) compares to the form of the axisymmetric incremental stiffness matrix for the continuum elements, equation (3.8), and the axisymmetric membrane elements, equation (4.22). The distortion rate matrix is obtained by the combination of two terms (c.f. equation (3.10)):

\[ [C] \, U = ([B]^T J) \, \sigma \, r + [B]^T \, \sigma \, \dot{r} \, J \]  \hspace{1cm} (5.54)

where the first term in the distortion rate matrix gives the rate of change of element geometry and is evaluated by algebraic manipulation, rather than by the term-by-term expansion procedure that was necessary for the axisymmetric membrane element distortion rate matrix (see Section 4.2.2). The second term of the \([C]\) matrix accounts for the rate of change of the nodal co-ordinates with time.

The first part of the distortion rate matrix may be rewritten by substituting for \([B]\) using equation (5.24), giving:

\[ ([B]^T J) \, \sigma \, r = [N]^T [L_1]^T ([S]^T J) \, \sigma \, r \]  \hspace{1cm} (5.55)

From equation (5.22):

\[ [S]^T J = \begin{bmatrix} r' & -z' \\ z' & r' \end{bmatrix} \]  \hspace{1cm} (5.56)
This equation may be differentiated with respect to time and multiplied by the stress vector \( \sigma \) to give:-

\[
([S]^T J) \sigma \quad = \quad \begin{bmatrix} \dot{r}' & -\dot{z}' \\ \dot{z}' & \dot{r}' \end{bmatrix} \begin{bmatrix} \tau \\ \sigma \end{bmatrix} \quad (5.57)
\]

which is re-arranged to give:-

\[
([S]^T J) \sigma \quad = \quad \begin{bmatrix} \tau & -\sigma \\ \sigma & \tau \end{bmatrix} \begin{bmatrix} \dot{r}' \\ \dot{z}' \end{bmatrix} \quad (5.58)
\]

The two variables, \( r \) and \( z \), are the co-ordinates of the stress point and may be conveniently written in the form:-

\[
\begin{bmatrix} r \\ z \end{bmatrix} \quad = \quad [A] r \quad (5.59)
\]

where \( r \) is the global co-ordinates vector as given in equation (5.11) and the matrix constant, \([A]\), is:-

\[
[A] \quad = \quad \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \quad (5.60)
\]

Differentiating equation (5.59) with respect to both \( \alpha \) and time, and then substituting \( u' \) using equation (5.12) gives:-

\[
\begin{bmatrix} \dot{r}' \\ \dot{z}' \end{bmatrix} \quad = \quad [A] u' \\
\quad = \quad [A] [N'] U \quad (5.61)
\]

Combining this expression with equation (5.58) gives:-

\[
([S]^T J) \sigma \quad = \quad [Q] [A] [N'] U \quad (5.62)
\]

5-24
where:

\[
\begin{bmatrix}
\tau \\
\sigma \\
\tau \\
\end{bmatrix}
\]  

(5.63)

If equation (5.62) is combined with equation (5.55), then the first term in the distortion rate equation (5.54) may be shown to be:-

\[
([B]^T J) \sigma r = [N]^T [L]^T [Q] \bar{A} [N^-]^T U r
\]

\[
= [C_i] \bar{r} 
\]

(5.64)

The second term of the \([C]\) matrix may be rewritten by substituting equation (5.49) and noting that \(\bar{K} = U\). The distortion rate matrix can therefore be written fully as:-

\[
[C] \bar{U} = [C_i] \bar{r} + [B]^T \bar{\sigma} [N] \bar{U} J
\]

(5.65)

A seven-point Gaussian quadrature scheme is used to evaluate the incremental element stiffness matrix of equation (5.52). As previously discussed in Section 4.2.2, exact integration can only be achieved for an element that has the nodes equally spaced, is initially straight and does not experience any severe distortion. If the finite element analysis was carried out as a small displacement problem, then the exact integration pre-requisites would be satisfied and it can be shown from Appendix 5A that a seven-point Gauss quadrature would be sufficient. However, within a large displacement analysis the interface element will be subjected to finite distortion and exact integration in this case is not possible. Therefore, it is reasonable to choose an order of quadrature based on the ideal configuration, although the limitations of this chosen integration scheme must be realised.

At the end of each step of the finite element analysis it is necessary to update each of the nodal co-ordinates of the deformed mesh by the calculated incremental displacements. For the interface elements this is done by determining the individual updated nodal co-ordinates and then calculating the arithmetic mean for the corresponding pairs and updating the co-ordinates to this mean position, so that they continue to share a common position.
Note that all of the plane strain large displacement computations described in this thesis were carried out using the six-noded plane strain interface element (Burd and Brocklehurst (1991)), with a three-point Gaussian integration rule.

5.4.5 Limitations of the Interface Element

The formulations of the axisymmetric and plane strain interface elements are based on an elastic perfectly-frictional model of behaviour. In modelling a two-layer soil system this interface element is appropriate for the boundary between the frictional fill layer and the reinforcement, but it is somewhat limited when modelling the clay-reinforcement boundary because of the cohesive nature of the clay.

The potential problem in modelling a cohesive interface with frictional interface elements is that the predicted shear stresses may exceed the shear strength of the clay (as seen in Figures 7.5, 8.2, 9.4, 10.2 and 10.3). This is because the discretization in the clay may allow nodal forces to exceed the clay strength. Although the interface element formulations could be broadened to include a cohesive strength term as well as friction, which would ensure that the shear stresses did not exceed the clay shear strength, this would make only a difference to the absolute peak shear stress values and no difference to the overall trend of the stress distributions.

The further limitation of the axisymmetric and plane strain interface element formulations is that they do not allow correctly for the re-joining of the mating surfaces after they have been relaxed. For instance, if the normal stress across the element was to become tensile (i.e. positive), thus tending to separate the interface surfaces, the formulation immediately sets the stresses to zero, but does not calculate the would-be separating distance. When the normal stress becomes compressive again, the formulation assumes the surfaces are at once in contact, rather than allowing for the potentially large separating distance to be recovered. Despite this shortcoming the interface element formulation is applicable to the modelling of the boundaries in the two-layer soil systems considered herein, because separation of the mating surfaces does not occur (see the contours of vertical stress, plates 6.3 and 6.4 Appendix 6B).
5.5 Verification of the Axisymmetric Interface Formulation

The new large strain axisymmetric interface formulation is validated by a series of test analyses. A lack of appropriate benchmarks has meant that a less than comprehensive validation for this element has been undertaken, i.e. all the tests are for horizontal and vertical configurations only. Some of these analyses were found initially to be rather unstable, so in order to improve the computation the incremental stiffness matrix \([K]\), equation (5.52) is made purely elastic by negating the plasticity part of the elasto-plastic material stiffness matrix \([D]^{ep}\). Although the element stiffness is then based upon the elastic stiffnesses only, the stress updating still considers plasticity and therefore the overall solution for stresses is correct. This approach has been used for all the interface element analyses discussed throughout this dissertation. A similar procedure was adopted also by Bell (1991).

5.5.1 Single Interface Element under Constant Normal Stress

A small strain analysis is conducted using a single axisymmetric interface element as shown in Figure 5.9, where the length of the element, \(l\), and the inner radius from the centre-line to the element are both taken as unity. The lower nodes of the interface are fixed horizontally and vertically, while the upper nodes are each displaced horizontally by the same amount, \(u_r\). A constant normal stress of \(\sigma = 100\ kPa\) is applied throughout the analysis.

The shear and normal stiffnesses of the interface, \(k_s\) and \(k_n\), are equal to \(10^4\ kN/m\) and \(10^8\ kN/m\) respectively, and the angles of friction and dilation of the interface are \(\phi_i = 30^\circ\) and \(\psi_i = 0^\circ\), respectively. The resulting shear stresses, \(\tau\), and shear displacements, \(u_r\), are depicted in Figure 5.9 for the case of initial loading, partial unloading and then re-loading. The stresses and displacements of the seven Gauss points are identical to each other and the correct behaviour is obtained.

5.5.2 Single Interface Element Subjected to Yield

A single horizontal axisymmetric interface element, similar to that shown in Figure 5.9, is fixed horizontally and vertically along the lower surface and subjected to a variable compressive normal stress, \(\sigma\), along the upper surface. By applying increments of horizontal
displacement to each of the upper nodes, the normal and shear stress paths of a single Gauss point can be traced, as shown in Figure 5.10 for different initial $\sigma$. The stresses and displacements of all of the integration points are identical. Once the displacements in each case become large enough to cause yield of the interface, further displacements generate both increasing $\sigma$, due to the assumed dilation of the interface, and $\tau$ as the stress paths correctly follow the yield surface. The interface properties are given in Figure 5.10.
5.5.3 Adjacent Concentric Thick Cylinders

Two concentric adjacent cylinders of wall thickness and inner radius \( a = 1 \, \text{m} \), separated vertically by interface elements, are subjected initially to a uniform internal outward acting pressure, \( q = 100 \, \text{kN/m} \), so that the interface is under compression. The outer cylinder is then vertically displaced relative to the inner cylinder by an amount \( \delta \), while holding \( q \) constant. The mesh of sixteen 15-noded axisymmetric continuum elements and two interface elements is shown schematically in Figure 5.11.

The cylinders are assigned a Young’s modulus of \( E = 5 \times 10^4 \, \text{kPa} \), a shear strength of \( 1.2 \times 10^9 \, \text{kPa} \) and a Poisson’s ratio of zero. This small displacement analysis is conducted for two different interface friction angles, \( \phi_i = 20^\circ \) and \( \phi_i = 30^\circ \), while the other interface properties of dilation angle, \( \psi_i = 0 \), and the shear and normal stiffnesses, \( k_s = 10^5 \, \text{kN/m} \) and \( k_n = 10^{10} \, \text{kN/m} \), are kept constant.

![Figure 5.11: Pressure against Displacement for Outer Thick Cylinder](image)

The pressure, \( P \), at the base of the outer cylinder is plotted in Figure 5.11 against the small relative displacement, \( \delta \). The limit pressure, \( P_u \), for each analysis is dependent upon the value of the friction angle and can be analytically determined by the equation:-

\[
P_u = 2a \, \sigma_i \tan \phi_i
\]  
(5.66)
where $\sigma_i$ is the radial stress across the interface (of length $2a$). Since the interface is of infinitesimal thickness and its stiffness does not significantly affect the radial stress distribution through the cylinders (as shown in Figure 5.13), $\sigma_i$ can be calculated by using the equations (4.53), which are derived by equilibrium, compatibility of strain and the stress-strain relationships of elasticity. By applying the boundary conditions (at $r = a$, $\sigma_r = -q$ and at $r = 3a$, $\sigma_r = 0$), it may be shown that at $r = 2a$:

$$\sigma_i = \frac{q}{8} \left( 1 - 9 \left( \frac{a}{r} \right)^2 \right) = -15.625$$  \hspace{1cm} (5.67)

There is a good agreement between the finite element solutions and the simplified analytical calculations for $P_u$, Figure 5.11. Although the average shear stress, $\tau$, along the interface gives the correct yield response for increasing $\frac{\delta}{a}$, Figure 5.12, there is a slight variation between the distributions of $\tau$ along the lengths of the higher and lower interface elements ($\pm 5\%$), due to the bending moment created by the displaced outer cylinder.

![Graph showing shear stress at interface against displacement](image)

**Figure 5.12: Shear Stress at Interface against Displacement**

For the initial loading condition, where only a uniform internal pressure $q$ is applied, the radial and circumferential stresses are continuous through the two cylinders, Figure 5.13, with the meridional stresses at zero. The finite element results are for each Gauss point throughout the continuum elements, with the analytical elasticity solutions shown for comparison. The results display an exact match for this small displacement elastic analysis.
5.5.4 Elastic Annular Plate under Inward Internal Pressure

An elastic annular plate of thickness and inner radius \( a = 1 \ m \) and outer radius \( b = 10a \), modelled by fourteen axisymmetric continuum elements, is sandwiched between two layers of interface elements as shown in Figure 5.14. A uniform vertical stress of \( q = 10 \ kN/m \) is applied continuously across the system, while the inner cavity is progressively displaced inwards by an amount \( \delta \), generating an inward internal pressure \( P \).

The properties assigned to the plate are a Young's modulus \( E = 10^5 \ kPa \), a shear strength of \( 1.2 \times 10^9 \ kPa \) and a Poisson's ratio of zero. The two identical interfaces have the properties of a friction angle \( \phi_i = 30^\circ \), a dilation angle \( \psi_i = 0 \) and the shear and normal stiffnesses \( k_s = k_n = 10^5 \ kN/m \).

The value of \( P \) is dependent upon the magnitudes of both the inner displacement, \( \delta \), and the outward acting body forces imposed by the frictional interface elements on the top and bottom of the plate, \( F \) (per unit area). However, the magnitude of \( F \) (\( = \tau \)) is dependent upon \( \delta \) also, as shown in Figure 5.15. As the displacements increase at the inner radius \( (r = a) \) the shear stresses along the interfaces vary from the initial fully elastic case \( (\tau' = k_s \ u_r) \), to the partially plastic-elastic case (illustrated in Figure 5.15) and ultimately to the fully plastic case \( (\tau' = q \ \tan \phi_i) \). The complete analytical solution relating \( P \) and \( \delta \) for
the elastic-plastic interfaces proved unobtainable, but by assuming that only the fully plastic case prevails (i.e. \( F = q \tan \phi_c \)) and that the plate remains elastic, the following solution is derived.

\[ \frac{d^2 \delta}{dr^2} + \frac{1}{r} \frac{d \delta}{dr} - \frac{\delta}{r^2} = -\frac{2F}{E} (1 - \nu^2) \]  \hspace{1cm} (5.68)

The sum of the complementary function and the particular integral of this second order linear differential equation gives:

\[ \delta = A \frac{r}{2} - \frac{2F r^2}{3E} (1 - \nu^2) \]  \hspace{1cm} (5.69)

where \( A \) and \( B \) are constants determined from the boundary conditions, i.e. at \( r = a, \sigma_r = P \) and at \( r = b, \sigma_r = 0 \).
Solving for $A$ and $B$ gives:

$$P = C \left\{ \delta + \frac{2F}{3E} \left[ (1-v^2)r^2 - \frac{(2+v)}{(a+b)r} [(b^2 + ab + a^2)(1-v)r^2 + a^2b^2(1+v)] \right] \right\}$$ \hspace{1cm} (5.70)

where the constant $C$ is:

$$C = \frac{E (a^2 - b^2) r}{a^2 [r^2 (1-v) + b^2 (1+v)]}$$ \hspace{1cm} (5.71)

The correlation between the finite element solution and the simplified analytical calculation for $P$, presented in Figure 5.14, is very close. There is however an observable discrepancy between the results at small values of $\delta$, because, contrary to the analytical assumption, full plasticity has not developed across the entire interface lengths, Figure 5.15.

![Figure 5.15: Shear Stress Distribution along Interfaces](image-url)
APPENDIX 5A

Finite Element Equations for the Ten-Noded Interface Element

Figure 5.16 shows the isoparametric mapping between the parent and reference elements for the ten-noded interface element.

![Graph showing isoparametric mapping between parent and mapped elements.]

Parent Element  
Mapped Element

Figure 5.16: Mapping for the Ten-Noded Axisymmetric Interface Element

The vectors \( r \) and \( R \), that are related by equation (5.11), are defined as:

\[
\begin{aligned}
r^T &= \begin{bmatrix} r_a & z_a & r_b & z_b \end{bmatrix} \\
R^T &= \begin{bmatrix} R_1 & Z_1 & R_2 & Z_2 & \ldots & R_{10} & Z_{10} \end{bmatrix}
\end{aligned}
\]  
(5.72)  
(5.73)

Where \( R_i \), \( Z_i \) are the global co-ordinates of the \( i^{th} \) node. The shape function matrix, which is required to keep the co-ordinates \( (r_a, z_a) \) and \( (r_b, z_b) \) congruent, is of the form:

\[
[N] = \begin{bmatrix}
f_1 & 0 & f_2 & 0 & \ldots & f_5 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & f_1 & 0 & f_2 & \ldots & 0 & f_5 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & f_1 & 0 & f_2 & \ldots & f_5 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & f_1 & 0 & f_2 & \ldots & 0 & f_5
\end{bmatrix}
\]  
(5.74)
in which:

\[ f_1 = \frac{2}{3} (\alpha + 1/2) (\alpha) (\alpha - 1/2) (\alpha - 1) \]  (5.75)

\[ f_2 = \frac{2}{3} (\alpha + 1/2) (\alpha) (\alpha - 1/2) (\alpha + 1) \]  (5.76)

\[ f_3 = -\frac{8}{3} (\alpha + 1) (\alpha) (\alpha - 1/2) (\alpha - 1) \]  (5.77)

\[ f_4 = 4 (\alpha + 1) (\alpha - 1) (\alpha + 1/2) (\alpha - 1/2) \]  (5.78)

\[ f_5 = -\frac{8}{3} (\alpha + 1) (\alpha) (\alpha + 1/2) (\alpha - 1) \]  (5.79)

Note, these shape functions are the same as those used for the five-noded axisymmetric membrane element.

The matrix \([B]\) of equation (5.24) is given by:

\[
[B] = \frac{1}{J} \begin{bmatrix}
-f_1 r' & -f_1 z' & \cdots & -f_5 r' & -f_5 z' & f_1 r' & f_1 z' & \cdots & f_5 r' & f_5 z'
\end{bmatrix} 
\]  (5.80)
CHAPTER 6:

INVESTIGATION OF PLANE STRAIN AND AXISYMMETRY

6.1 Introduction

The purpose of the finite element parametric study, undertaken in Chapters 7 - 10, is to investigate the reinforcement mechanisms operating in a two-layer soil system, with and without the inclusion of a membrane possessing a medium angle of friction ($\phi_i = 35^\circ$), and the influence upon those mechanisms of certain material properties. The effects upon the performance of the two-layer soil system of including a low friction membrane ($\phi_i = 10^\circ$) are investigated in Chapter 11. The key areas of investigation are, (a) the load-displacement responses, (b) the stresses acting at the fill-clay, or fill-reinforcement-clay, interfaces and (c) the reinforcement tensions.

The important material properties of a two-layer soil system, which are varied in the parametric study, are the thickness and strength of the fill layer (Chapter 7), the reinforcement stiffness (Chapter 8), the reinforcement length (Chapter 9) and the clay shear strength (Chapter 10). The majority of these parametric studies are conducted for plane strain conditions only, with the notable exception of the investigation of the influence of the clay strength (Chapter 10) which is for plane strain and axisymmetry. To conduct a thorough parametric investigation it is first necessary to establish the most appropriate numerical approach. These preliminary plane strain studies are described in Sections 6.2 and 6.3.
Utilising the optimum numerical approach established in the first part of this chapter, Section 6.4 proceeds to compare the results of the three key areas for a plane strain and a corresponding axisymmetric analysis for the unreinforced and reinforced case, using the newly developed elements from Chapters 4 and 5. A detailed investigation of the displacements, strains, stress distributions, principal stress directions in the fill, the mobilised friction angle and the stress distributions on the underside of the footing, for the plane strain and axisymmetric reinforced and unreinforced 'central parametric analyses' is undertaken in Section 6.5.

6.2 Specification of the Parametric Study

The philosophy that is adopted throughout this study is to investigate thoroughly the properties of a two-layer soil system by examining the monotonic load behaviour of a reinforced and unreinforced structure. The influence and effects of certain specific reinforcement and soil parameters are then studied carefully by varying those parameters and conducting investigations of the three key areas listed earlier.

The parametric study of two-layer soil systems is conducted with particular consideration for their practical application as reinforced and unreinforced unpaved roads, since this is the most researched use of two-layer soil systems, but the study is also very relevant to all other uses of such structures. The study is accomplished by considering a 'typical two-layer soil system', schematically shown in Figure 6.1, as a central analysis, which is allocated both a specific geometry and a set of typical material properties, which are varied individually.

The central analysis of the 'typical two-layer soil system' is established by considering the conventional design of unpaved roads and the characteristic properties of the reinforcement, the aggregate subbase and clay subgrade usually found at the sites of unpaved reinforced roads, as well as the average wheel widths of the lorries used to traffic such roads. These central analysis parameters are all given in Appendix 6A. The soil properties are intended to be representative of a typical reinforced unpaved road built on a clay subgrade of medium strength, therefore an undrained shear strength of $\text{s}^{PS}_u = 30 \text{kPa}$ (plane strain) is
Figure 6.1: The ‘Typical Two-Layer Soil System’

chosen, which is classified as ‘soft’ in British Standard CP 2004 (1972). The aggregate fill generally used for unpaved roads is a Type 1, or occasionally Type 2, Subbase Material (see Specification for Highway Works (1986)), which has a mean peak friction angle of between $\phi_{peak} = 46^\circ \rightarrow 56^\circ$, and it is frequently laid at an average depth of around $D_f = 300 \text{ mm} \pm 100 \text{ mm}$ for such applications as on-site haul roads. Geosynthetic reinforcements are manufactured with a wide range of material stiffnesses (in the order of between $J = 30 \rightarrow 2000 \text{ kN/m}$ approximately), but a proprietary polymeric geogrid material would typically have a stiffness of $300 \text{ kN/m}$. Finally, from investigations of previous unpaved road trials, (e.g. the full-scale reinforced haul road test at the Science and Engineering Research Council owned Soft Clay Site at Bothkennar, Little (1992), and the Transport and Road Research Laboratory tests, Pike et al. (1977)), the typical overall width of the double-tyred wheel on the lorries used, is found to vary only slightly from between $0.51 \text{ m} \rightarrow 0.47 \text{ m}$. Therefore a wheel (or footing) width of $0.5 \text{ m} = 2B$, with a perfectly rough contact, is adopted throughout this research. Although there are, obviously, two sets of double-tyred wheels, one at each end of the lorry axle, they are usually sufficiently widely spaced so that they may be regarded as independent footings for design purposes. It is also
worth noting that although the wheels are generally double-tyred (as shown in Figure 1.1),
the loading is considered as a single wheel load since the separation of the tyres is effectively
negligible.

The finite element analyses are all displacement controlled so that a constant final
vertical footing displacement of $\delta = 0.15 \ m = 0.6 \ B$ is obtained in each finite element run.
At this vertical displacement the associated rut that would develop in an unpaved road is
generally considered to be too deep for the minimum clearance required by the majority of
lorry traffic. If trafficking becomes impossible then the unpaved road has effectively failed,
therefore the footing pressure required to achieve this displacement will be considered as
the ultimate (or final) footing pressure $P_f$, for that particular road.

Since the unpaved road is symmetrical, it is only necessary to consider the half of the
problem about a centre-line through the footing load, provided that appropriate constraints
are employed along the finite element mesh centre-line, as shown in Figure 6.1. That is,
fixity in the horizontal 'x' direction and freedom in the vertical 'y' direction, i.e. a roller
type boundary.

6.3 Experiments with Preliminary Plane Strain Meshes

To run a finite element parametric study it is essential to use a mesh of an appropriate size
with an element density that is optimum, in terms of computational efficiency and numerical
accuracy, which is discussed in Sections 6.3.1 and 6.3.2 respectively. The influence of the
mesh boundaries is assessed in Section 6.3.3, and Section 6.3.4 investigates the differences
in calculated behaviour between a large and small displacement analysis. The various
reinforcement mechanisms acting at large and small displacements are identified in Section
6.3.5.

6.3.1 Mesh Size

Initially it was assumed that the size of the finite element mesh needs to be only sufficiently
large to contain the size of the clay plasticity zones as predicted by plasticity theory,
illustrated in Figure 6.2. A similar assumption was made by Poran (1985). The load distribution through the fill is simplified to a uniform pyramidal distribution given by a constant load spread angle, $\beta$, as described in Section 6.3.5.

Giroud and Noiray (1981) suggested that, according to the theory of plasticity (as explained in Calladine (1985) and Bolton (1979)), the overall width of the plastic failure zone ($W_p$ shown in Figure 6.2) in a frictionless incompressible subgrade underlying a geotextile reinforced subbase is:

$$W_p = 3 (2B + 2D_f \tan \beta)$$  \hspace{1cm} (6.1)

where $B$, $D_f$, and $\beta$ are the same as illustrated in Figure 6.1.

![Figure 6.2: Plastic Failure Mechanism in Cohesive Material
(After Bolton (1979))](image)

This width ($W_p$) serves as a guide-line for the required width of the finite element mesh. The minimum depth of the plastic failure zone in the clay ($H_p$ shown in Figure 6.2), is:

$$H_p = (B + D_f \tan \beta) \sqrt{2}$$  \hspace{1cm} (6.2)
Similarly, this depth serves as a guide-line for the minimum required depth of the clay. The overall minimum depth of the mesh is the sum of the thicknesses of the subbase and the clay (i.e. $D_J + H_P$).

From approximate and exact load distributions given by Perloff and Baron (1976) it is suggested that a load spread angle, such as $\beta$ in Figure 6.1, should be taken as $\beta = \tan^{-1} 0.5 \approx 26.6^\circ$. However, Taylor (1965) suggests that a load spread angle of $30^\circ$ is still conservative. Poran (1985) estimated from plate bearing test results and from the finite element analyses of those tests, that an angle of the order of $45^\circ$ was appropriate. A comprehensive list of the recommended load spread angles by other authors is presented in Craig and Chua (1990). From the plane strain finite element results of this dissertation it is estimated, by the method described in Section 6.3.5, to be an average value of about $40^\circ$ for the particular material properties chosen (Appendix 6A). Using $\beta = 40^\circ$, $B = 0.25 \text{ m}$ and $D_J = 0.3 \text{ m}$ in equations (6.1) and (6.2), the respective values of $W_p$ and $H_P$ are determined and hence the minimum required mesh size:

\[
\text{The minimum mesh depth} = D_J + H_P = 1.2B + 2.84B = 4.04B
\]

\[
\text{The minimum mesh half width} = W_p / 2 = 6.0B
\]

At the base of the mesh it is felt that the boundary should be removed just beyond the minimum clay depth ($H_P$), in order to allay the amount of disturbance that any edge effects may cause to the predicted plastic zone. Therefore, the overall depth of the mesh is $D_J + D_e = 1.2B + 3.2B = 4.4B$. Additionally, a roller type boundary is used at the base similar to Burd (1986), rather than complete fixity as used by both Poran (1985) and Zeevaert (1980), since it is felt that there may still be some significant horizontal and shear stresses present. Although this introduces an artificial plane of weakness, it is considered sufficiently removed so as not to effect the results. It is assumed that another roller boundary condition at the minimum mesh half width distance of $(W_p / 2)$ from the centre-line, is remote enough to prevent horizontal and shear stress concentrations occurring. As discussed in Section 6.2 the problem is symmetrical about the footing centre-line.
For comparison the finite element mesh sizes employed in other studies of two-layer soil systems, related to the relevant footing half widths (or radii), $B$, used in that study, are:

Zeevaert (1980): $W_e = 6.0 \, B$ and $D_e = 4.5 \, B$ for $D_f = 1.5 \, B$, $2.33 \, B$

Griffiths (1982): $W_e = 9.2 \, B$ and $D_e = 3.2 \, B$ for $D_f = 0.8 \, B$

Poran (1985): $W_e = 6.75 \, D_f$ and $D_e = 4.5 \, D_f$ for $D_f = 0.4 \, B$, $1.0 \, B$, $2.0 \, B$

Burd (1986): $W_e = 13.33 \, B$ and $D_e = 10.66 \, B$ for $D_f = 1.33 \, B$, $2.0 \, B$, $2.67 \, B$

Herein: $W_e = 6.0 \, B$ and $D_e = 3.2 \, B$ for $D_f = 0.6 \, B$, $1.2 \, B$, $2.0 \, B$

The research conducted herein to examine the influence of the fill depth (Chapter 7) adopts the same philosophy as used by Zeevaert (1980) and Burd (1986), in that one constant set of dimensions are used for the size of the clay part of the mesh and for the footing width, while the fill depth is varied. Poran (1985) however, kept all the soil dimensions constant and varied the footing widths.

Using this mesh a series of plane strain finite element investigations are undertaken to determine the effects of element density and the mesh boundaries.

6.3.2 Mesh Density

The finite element method of analysis hinges on the process of being able to subdivide the entire problem into a finite number of representative components, or discrete parts, whose behaviour is readily understood. By ‘rebuilding’ the original system from these elements the behaviour of the whole system can then be studied. However, there are certain approximations and idealizations adopted for each element, which consequently cause slight inexact estimations. These discretization errors and other inherent inaccuracies, such as round-off, may prevent the calculation from converging to the correct result. It is generally accepted (Zienkiewicz (1977)) that the numerical approximation can be systematically improved to approach the exact answer by decreasing the size of the elements and hence increasing the total number of parameters specified at nodes.

The amount of distortion experienced by the elements during the large strain calculation can also have a detrimental influence on the accuracy of the numerical solution. Burd (1986) suggests that this difficulty may be dealt with either by refining the mesh in the region of
distortion, or by using a higher order quadrature rule for those elements that tend to become heavily deformed. Although Burd (1986) recommends the latter approach, the following investigative study adopts the former suggestion of improved refinement to try and alleviate numerical inaccuracies caused by unfavourable distortion effects, notice the large number of elements at the footing edge in Figure 6.3.

The consequences of varying the element sizes are investigated experimentally using the original mesh geometry established in Section 6.3.1 and a constant topography of element zones within the mesh. By maintaining a consistent mesh geometry, but increasing the total number of elements in the mesh, the density of elements increases and therefore the element sizes decrease.

The plane strain meshes for the four different numbers of elements (105, 240, 415 and 1000 elements) are shown in Figure 6.3. Each of the four finite element meshes, created using the mesh generator program OXMESH (Houlsby (1988)), are prescribed exactly the same material properties for the fill, clay and reinforcement, and subjected to the same final footing displacement of 0.6 $B$ in similar increment numbers (either 150 or 210 steps). This particular set of finite element analyses pre-dates the development of the plane strain interface elements.

The intention is to conduct a convergence study of the large displacement plane strain finite element results obtained for an increasing number of mesh elements and to effectively undertake a sensitivity study of how much these results are influenced by increasing the mesh resolution from 'coarse', Mesh P1, to 'fine', Mesh P4. The comparisons of the results (i.e. the load-displacement responses, the calculated stresses acting on the reinforcement and the reinforcement tensions at the footing centre-line) obtained for each mesh size are made with respect to the results of Mesh P4, since it is assumed that these results will be the closest to the exact answer given the practical constraints of computing time.

The pressure-displacement responses for each of the three coarser meshes (Mesh P1, P2 and P3) are compared to the response of Mesh P4 at each increment of footing displacement. Since the finite element calculations are done in either 150, or 210 increments of footing displacement, a linear interpolation is needed to establish values of footing
Figure 6.3: Four Finite Element Meshes with Varying Element Densities

pressure, from the Mesh P4 response, at all the various values of displacement used in the other three meshes. A total of the absolute percentage difference between each value of footing pressure for each run, compared to Mesh P4, is determined and averaged across the number of data. This, then, gives an average percentage error of the pressure-displacement response for each of the three coarser meshes in comparison to Mesh P4. These calculations of error are plotted in Figure 6.4.
The distribution of both shear and normal stress, above and below the reinforcement length, is calculated by Method 4 as described in Section 5.2. The absolute areas contained beneath these stress profiles for the coarser meshes are compared to the areas under the Mesh P4 stress profiles. The areas under the normal stress profiles are equal to the normal forces acting at the fill base and clay surface, while the areas under the shear stress profiles are equal to the shear forces exerted by the clay and the fill on the reinforcement. The percentage difference in these forces for each coarse mesh, with respect to Mesh P4, is also shown in Figure 6.4.

A distribution of the reinforcement tension along the reinforcement length is obtained by calculating the average of the three Gauss point stresses, per plane strain membrane element, plotted at the average distance of those Gauss points from the footing centre-line. The average value of stress is used because of the minor oscillations that occur in the tension distribution when each individual Gauss point stress is plotted. The reinforcement tension at the centre-line is deduced by linear interpolation from the averaged values of the tension for the first two membrane elements. The percentage differences in the centre-line tension, for each mesh with respect to Mesh P4, are shown in Figure 6.4.

![Graph showing percentage error with respect to Mesh P4 of the Finite Element Results](image)

**Figure 6.4: Percentage Error with Respect to Mesh P4 of the Finite Element Results**
Figure 6.4 shows a series of six lines representing the percentage difference between the results obtained with Meshes P1, P2 and P3 compared to those obtained with Mesh P4. The assumption is made that any inherent error in the results provided by Mesh P4 is of an insignificant amount, implying that the percentage difference between the coarser meshes and Mesh P4 is an indication of the percentage error associated with those larger element sized meshes. The noticeably different trend of the shear fill force line, compared to the other five lines in Figure 6.4, is due to the spuriously low value of error estimated for Mesh P1 and the unexpectedly high value estimated for Mesh P3.

For Mesh P1 the errors vary considerably in each of the different calculations, with a maximum error of 25.1% obtained in the calculation of the shear force on the clay. Clearly, there are significant errors that develop in a finite element analysis which uses such a 'coarse' mesh.

For Mesh P2 the maximum error is obtained in the calculation of the shear force at the fill base, 15.8%. Although errors are also found in the estimations of the shear force at the clay surface and the reinforcement tension at the centre-line, approximately 10%. However, very small errors occurred in the prediction of the two normal forces and the load-displacement response, around 1.3% → 3.3%.

For Mesh P3 the majority of the errors are within just 3.4%, except for the shear force at the fill base which is 15.7% different from the shear force calculated using Mesh P4.

With regards to minimising any discretization errors and achieving a good approximation to the exact solution, then clearly the mesh to use would be Mesh P4. However, this fine mesh requires great amounts of computing time, which renders it almost useless as a practical design tool. Mesh P3, although less than half the number of elements compared to the fine mesh, still needed around 60% of the same computing time, and additionally had an average inherent error of about 4%. Therefore, it is decided that the optimum mesh size is Mesh P2, since this gives answers with an average percentage error of about 7%, but requires only 20% of the computing time compared to the fine Mesh P4. This error is considered as acceptable, unlike that for Mesh P1 which has an average
percentage error of 13% while still requiring 12% of the Mesh P4 computing time. Thus, for accuracy and efficiency Mesh P2 is used for the initial finite element calculations that are made with the mesh of clay dimensions $6B \times 3.2B$.

6.3.3 Influence of Mesh Boundaries

The influence of the proximity of the mesh boundaries to the footing, on the results obtained from a plane strain large displacement finite element analysis, are investigated in this section.

The mesh dimensions for the clay part of the two-layer soil system were initially established in Section 6.3.1 as $W_c = W_p / 2 = 6B$ and $D_c = 3.2B$. However, a comparative analysis is conducted using a plane strain mesh with the clay dimensions of $W_c = 20B$ and $D_c = 20B$, and a fill thickness $D_f = 1.2B$ (illustrated in Figure 5.2), for the same two-layer soil system material properties given in Appendix 6A. This larger mesh is developed from the 240-element Mesh P2, shown in Figure 6.3, by increasing the area of the outer zone and adding in more elements, but maintaining the same element aspect ratio. Consequently the total number of triangular continuum elements increases from 240 (with 515 nodes) to 449 (with 1036 nodes). The $6B \times 3.2B$ mesh was generated before the implementation of the plane strain interface elements, whereas the $20B \times 20B$ mesh does possess interface elements at the fill-reinforcement and clay-reinforcement boundaries (as do all the other analyses reported in this chapter).

The results show that the clay subgrade actually yields outside of the theoretically predicted failure zones, such that the real failure depths are $H_p = 6.5B$ and $8.6B$ for the unreinforced and reinforced cases respectively (as opposed to $2.84B$) and the real failure half widths are $W_p / 2 = 4.4B$ and $6.3B$ (as opposed to $6.0B$). The true failure zone is clearly seen in Figure 6.5, which plots all the yielded Gauss points throughout the two-layer soil system at the final footing displacement $\delta = 0.6B$. The original mesh size is shown in Figure 6.5 to be well within the clay failure zone, particularly for the reinforced case, which demonstrates the need to select the mesh dimensions carefully.
Figure 6.5: Yielded Gauss Points for the 20B X 20B Mesh B at $\delta = 0.6B$

Note that the horizontal, vertical and shear stresses that the fill layer experiences are sufficient to cause yield throughout its length in these particular plane strain analyses. This is discussed in more detail in Section 6.5.6.

Some of the consequences of using a larger mesh are that the initial elastic domain of the reinforced load-displacement response is approximately half as stiff as that for the smaller mesh and that the ultimate footing pressure, $P_f$, at a final displacement of $\delta = 0.6B$ is about 25% greater, as shown in Figure 6.6, where $P / s_{ps}$ is the average value of vertical stress beneath the footing and $\delta / B$ is the vertical footing displacement.

Additionally, the normal stresses, $\sigma$, acting directly under the footing on the upper and lower reinforcement surfaces, Figure 6.7, are larger for the 20B X 20B mesh (extracted directly from the interface elements) than for the 6B X 3.2B mesh (obtained by Method 4, Section 5.2). At the centre-line ($x = 0$) the normal stresses are approximately 43% greater for the larger mesh compared to the smaller one. However as $x$ increases the difference between these normal stresses reduces, such that at $x = 5B$ the difference in the estimated applied forces on the upper reinforcement surface (i.e. the areas under the upper normal stress curves between $x = 0 \rightarrow 5B$) is 8.6%, and similarly for the forces on the lower surface the difference is 11.1%.
Figure 6.6: Pressure-Displacement Response for the Two Mesh Sizes (20B X 20B and 6B X 3.2B), using the same Material Properties

Figure 6.7: Normal Stresses ($\sigma / s_{u}^{pS}$) on Upper and Lower Reinforcement Surfaces for the Two Mesh Sizes, using the same Material Properties

Although there are quite considerable differences in the normal stress distributions for the large and small meshes, there is less of a difference between the reinforced shear stress

6-14
distributions, \( \tau \), Figure 6.8. The upper shear stress profiles follow an almost identical pattern, whereas the lower shear stress profiles do vary slightly, especially directly under the footing where the small mesh tends to over-predict the magnitude of the stresses.

![Graph showing shear stresses comparison](image)

**Figure 6.8: Shear Stresses (\( \tau / s_u^{ps} \)) on Upper and Lower Reinforcement Surfaces for the Two Mesh Sizes, using the same Material Properties**

The tension, \( F_r \), along the length of the reinforcement, \( x \), is shown in Figure 6.9. The larger mesh predicts a reinforcement tension at the centre-line which is 20% greater than the smaller mesh prediction.

Clearly the proximity of the two smooth boundaries to the footing have a considerable effect on the finite element results obtained. It is thought that the 20B X 20B mesh is likely to be larger than necessary, since an optimum mesh of \( W_e = 10B \) and \( D_e = 12B \), for the particular fill depth of \( D_f = 1.2B \), should be sufficient to contain the real failure zone within the clay. Nevertheless, the parametric study is undertaken with a mesh of clay dimensions 20B X 20B, so that the solutions produced by the finite element analysis can be considered reliable and precise.
Figure 6.9: Reinforcement Tension \( (F_r / (B \cdot s_u^{PS})) \) for the Two Mesh Sizes, using the same Material Properties

6.3.4 Large and Small Displacement Analysis

Since the nature of an unpaved road allows large ruts to develop, it is argued that any numerical modelling of such structures should be based on large displacement, large strain theory so that all the deformations and mechanisms acting in the road are accurately modelled. However, there are numerous applications of reinforced aggregate layers overlaying soft ground for which any significant surface rutting is undesirable, as discussed in Chapter 1. In order to investigate any mechanisms of reinforcement that may act in a reinforced two-layer soil system, where the particular application requires that the displacements at the surface of the fill layer should be small, a small displacement, small strain finite element analysis should be used.

The objective of this section and the next is to study the mechanisms of reinforcement that operate in a reinforced two-layer soil system when there is no surface, or membrane deformation and then to investigate those additional mechanisms which may operate when large deformations are allowed to develop. To achieve this a comparison is undertaken of the plane strain finite element results obtained from the small and large strain analyses of
both a reinforced and an unreinforced two-layer soil system. These numerical results will also be compared to those predicted by the Houslsby et al. (1989) plane strain analytical design method, as discussed in the literature review Section 2.5.1.

The difference between small and large strain finite element analyses, is that in small strain the mesh geometry is not updated as the calculation proceeds, whereas geometric modifications are undertaken at each increment of displacement in large strain. In practical terms this means that for small strain the geometry of the finite elements remain unchanged during the loading process and that first order, infinitesimal, linear strain approximations can be used. By using a large strain analysis however, all of the geometric non-linearities associated with large displacements will be modelled accurately (as discussed in Section 3.2). The advantage of studying a small strain analysis is that the reinforcement mechanisms, that do not depend on geometric effects, become clearly visible. The ‘tensioned membrane effect’ for instance, is therefore explicitly excluded.

All of the research described in this section has been carried out using the finite element program OXFEM. Although this program is designed to conduct large displacement large strain calculations it has the facility to also carry out small displacement small strain analyses by simply by-passing the relevant large strain part of the program.

Four finite element runs are considered for this particular study of the different results obtained from large and small strain analyses of a reinforced and unreinforced two-layer soil system. The two-layer soil system dimensions and properties described in Appendix 6A are used with the large (20B X 20B) finite element mesh ‘B’, Figure 5.2. The system of referencing the plane strain finite element runs follows an order where the first letter gives the relevant mesh size (i.e. A, B or C, depending upon the particular fill depth \( D_f \)), the proceeding number refers to the fill friction angle in plane strain (i.e. 30°, 40° or 55°) and the final letters indicate whether reinforcement is present (‘R’) or not, and whether it is a small strain (‘S’) or large strain analysis, e.g. B40R is a large strain reinforced finite element run using mesh B, with \( \phi_{ps} = 40^\circ \). The axisymmetric finite element runs are proceeded by ‘AX’. For the corresponding Houslsby et al. (1989) and Houslsby and Jewell (1990) analytical
results, the referencing is generally the same as for the finite element runs, but preceded by an 'R', e.g. RB40R. Some slight variations of this referencing scheme do occur in this dissertation, but these are self-evident.

A comparison of the footing load-displacement responses for each of the four finite element runs and the four corresponding analytical results is shown in Figure 6.10, where $P$ is the average value of vertical stress beneath the footing and $\delta$ is the vertical footing displacement.

![Pressure-Displacement Responses](image)

**Figure 6.10: Pressure-Displacement Responses**

By determining the value of a load spread angle, $\beta$, (as described in Section 6.3.5) and an angle of friction on the footing base, $\delta_f$, (as described in Section 6.5.7) and using it in the Houlsby et al. (1989) method, along with the plane strain friction angle and clay strength, a direct comparison can be made of the results predicted by this analytical method and the calculated finite element results, i.e. the ultimate footing load ($P_f$) and the reinforcement tension ($F_r$).

Table 6.1 gives the run references and the final normalised footing pressures ($P_f/s_{u}^{PS}$) at a footing displacement of $\delta = 0.6B$ taken from Figure 6.10, as well as the calculated load.
spread angles (β) and angles of footing base friction (δf). The values of δf are negative because the shear stresses at the footing-fill boundary are found to be outward acting. The ‘R’ denotes a reinforced analysis and ‘U’ an unreinforced analysis.

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<tr>
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<th>F.E.</th>
<th>Analytical</th>
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<tr>
<td></td>
<td>δf</td>
<td>β</td>
</tr>
<tr>
<td>Large</td>
<td>R</td>
<td>-10.3°</td>
</tr>
<tr>
<td>Strain</td>
<td>U</td>
<td>-8.0°</td>
</tr>
<tr>
<td>Small</td>
<td>R</td>
<td>-3.0°</td>
</tr>
<tr>
<td>Strain</td>
<td>U</td>
<td>-1.4°</td>
</tr>
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**Table 6.1: Run References for the Finite Element and Analytical Results shown in Figure 6.10**

It is clear that the numerical load-displacement responses of the two large and two small strain finite element analyses are indistinguishable during the initial elastic loading, but that beyond a footing displacement of approximately 0.09 B the curves begin to diverge significantly. This suggests that the presence of the reinforcement has no significant influence on the performance of the two-layer soil system within the elastic regime, but plays more of an important role once the system becomes subjected to plastic deformations, as found by Love (1984).

For the small strain runs, B40S and B40RS, the improvement in the load capacity of the structure by including reinforcement is about 25%, at the final footing displacement of 0.6 B, while for the equivalent large strain analyses the improvement between runs B40 and B40R is about 30%. This shows that the reinforcement has a greater effect in improving the bearing capacity of the two-layer soil system for a large displacement, large strain analysis than in a small strain one (as also shown by Burd and Brocklehurst (1991)).
The amount of strengthening due to the purely small displacement reinforcement mechanism of the 'shear stress effect', evident in Figures 6.14 and 6.15, can be measured directly from the difference between the small strain unreinforced and reinforced runs, i.e. a 25% better load capacity at \( \delta = 0.6B \). The amount of additional improvement due to the large displacement reinforcement mechanisms, which develop as a result of the substantial rutting in the large strain analysis, can be estimated as the percentage difference between the final footing pressures for the small and large strain reinforced analyses, B40RS and B40R, i.e. approximately a 23% increase in bearing capacity. However, this 23% improvement can be attributed to both the influence of soil heave, which creates a surcharge and hence makes the entire system stiffer, as well as the reinforcement mechanisms of 'restraint' and 'tensioned membrane'.

The heave at the fill surface in a large displacement analysis creates an apparent improvement in the performance of the system. To calculate the true loading improvement obtained just through the purely large displacement reinforcement effects, the influence of any soil heave has to be removed. This can be quantified as the difference between the final footing loads for the small and large strain unreinforced analyses, B40S and B40, i.e. about 18%, therefore:-

<table>
<thead>
<tr>
<th>Reinforced large strain improvement</th>
<th>Unreinforced large strain improvement</th>
<th>Tensioned membrane and restraint improvement</th>
</tr>
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<tbody>
<tr>
<td>23 %</td>
<td>18 %</td>
<td>5 %</td>
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This effectively eliminates the surcharge effects at the surface for the large strain analyses.

This comparison illustrates well the need to use a large strain, large displacement formulation in the analysis of such two-layer soil systems if a complete finite element study of the reinforcement mechanism is to be conducted. Clearly, the geometric non-linear terms included in the finite element equations of a large strain analysis are necessary in order to capture all of the mechanisms of reinforcement. However, there is obviously a very important reinforcement mechanism acting in a two-layer soil system that does not rely on large
displacement effects, i.e. the 'shear stress mechanism'. There is also a comparatively minor 'improved load spread' of an extra 2° and 5° for these particular small and large strain analyses respectively, but this is not a general trend (as exhibited in Table 7.2).

Figure 6.10 also shows the ultimate footing loads predicted by the Houlsby et al. (1989) design method for the unreinforced systems (RB40 and RB40S) and the reinforced systems (RB40R and RB40RS). Although the analytical method is strictly a small displacement analysis its predictions can be compared with the results obtained from both the large and small displacement finite element solutions by using the appropriate values of β and δf, as given in Table 6.1.

Considering the small strain analyses first, it is evident that the Houlsby et al. (1989) method slightly overestimates the load capacity for both the reinforced and unreinforced systems, as predicted by the finite element analysis at a final footing displacement of δ = 0.6 B, (approximately 16% too high). Nevertheless the percentage of improvement between RB40S and RB40RS, approximately 25%, is equal to the final improvement between runs B40S and B40RS. For the large strain analyses, however, the analytical prediction of Pf for the reinforced system, RB40R, is a very good agreement to the finite element solution (approximately 3% too low), whereas the unreinforced prediction, RB40, is substantially lower than run B40 (21% less) because of the significant outward acting frictional force on the footing base (δf = −8.0°). The analytical method assumes that this force increases the magnitude of the outward shear stresses on the clay surface, thus further reducing the bearing capacity of the clay, but in fact the shear stresses are found to be inward acting in the area directly below the footing (Figure 6.14). Therefore, the analytical estimate of the percentage improvement between the unreinforced and reinforced systems for a large displacement analysis is about 60%, as opposed to the 30% improvement in the numerical calculations. Clearly the Houlsby et al. (1989) method can be considered a reasonably accurate design tool for estimating the ultimate load bearing capacity of reinforced and unreinforced two-layer soil systems when surface deformations are insignificant, but it is less applicable to the case where large displacements can occur.
6.3.5 Stresses acting on Reinforcement and Interpretation of Load Spread

In order to investigate the mechanisms of reinforcement in greater detail it is instructive to inspect the normal and shear stresses acting at the soil-reinforcement interface for both a small and a large strain finite element analysis. These stresses are extracted from the interface elements used in the finite element solution in the simple manner discussed in Chapter 5.

First consider the normal stresses (σ) developed along the interface of the soils, measured from the footing centre-line (x), at a final footing displacement of \( \delta = 0.6B \). Figure 6.11 shows the normal stresses acting at the unreinforced soil interface for the small and large strain finite element runs B40S and B40, and Figure 6.12 shows the corresponding small and large strain normal stresses that act when reinforcement is included, runs B40RS and B40R respectively.

Theoretical foundation design often assumes that imposed footing loads create a uniform pyramidal load distribution with depth through the soil and estimates are made of a constant load spread angle such as \( \beta \), see Giroud and Noiray (1981), Sowers et al. (1982), Poran (1985) and Housby et al. (1989). In reality there is unlikely to be a definite line distinguishing the extent of the load spread, rather, a zone across which the imposed footing pressure diminishes to zero, as seen in Figures 6.11, 6.12 and Plates 6.3 and 6.4. The limits of this zone would be extremely difficult to measure accurately in any model test. However, they can be established directly from the finite element results by studying the normal stress profile along the reinforcement length and determining the horizontal co-ordinates between which the distribution curve sharply decreases, indicating the edge of the area affected by the imposed load.

To simplify the problem it is assumed within this dissertation that, allowing for the self weight of the fill, the edge of the imposed loaded area at the fill base is defined by that horizontal distance from the footing centre-line \( (B') \) which contains 95% of the imposed normal footing stress \( (\sigma') \) on the upper reinforcement surface. In this way, the 95% limit gives a theoretical load spread angle \( \beta \). The process for determining this load spread angle is that the pressure due to the self weight of the fill is calculated and subtracted from the total normal stress at the fill base \( (\sigma) \) predicted by the finite element analysis. This will
then give the total imposed footing pressure along the reinforcement surface. A limiting point, from the footing centre-line, can be mathematically determined to give the exact cut-off for 95% of that full area under the normal stress curve influenced by the imposed footing load. This assumption means that a single value for the load spread angle can be obtained for any particular analysis.
Figure 6.11 actually shows four profiles of normal stress at the fill-clay interface, but the upper and lower stress distributions coincide for any particular run, because there is no reinforcement layer present. There is clearly a marked difference between the small strain and the large strain analyses within the region directly influenced by the footing load, the limit of which is shown by the load spread \( \beta_{pd0} \). The small and large strain, unreinforced, load spread angles (Table 6.1) would be almost at the same \( x \) co-ordinate, so only the large strain \( \beta \) value is plotted in Figure 6.11 and similarly for the reinforced case in Figure 6.12.

Considering the unreinforced analysis first the small strain normal stresses are at a maximum on the footing centre-line, where \( \sigma = 5.43 \, s_u^{PS} \), and they steadily decrease to the constant minimum surcharge, \( q \), of 0.19 \( s_u^{PS} \) which is due to the self weight of the fill \( (q = \gamma_f D_f) \). The large strain normal stresses start at \( \sigma = 5.54 \, s_u^{PS} \) on the centre-line and after increasing slightly to a maximum of 5.9 \( s_u^{PS} \) they then decrease until merging with the small strain results just beyond the load spread limit, at approximately \( x = 2.1 \, B \).

It is established from the upper and lower bound plasticity theorems developed for simple indentation problems in metal (Calladine (1985)), that the ‘exact’ solution for the ultimate bearing capacity \( (P_u) \) of an elastic, perfectly plastic cohesive material which is being subjected to a purely vertical bearing capacity type failure, with an applied surcharge of \( q \), is given by:

\[
P_u = N_c \, s_u^{PS} + q = (\pi + 2) \, s_u^{PS} + q \quad (6.3)
\]

where \( N_c \) is the bearing capacity factor and \( s_u \) is the undrained shear strength of the clay.

This plasticity theory predicts that the normal stresses transmitted onto the clay directly under the footing (Figure 6.11) should be at a constant value of \( (\pi + 2) \, s_u^{PS} + q = 5.33 \, s_u^{PS} \) within the load spread area, as shown schematically in Figure 6.13. Clearly the plane strain finite element computations of the normal stress do not give this theoretical constant distribution, which is because of the presence of some outward acting shear stresses (as discussed later), however the maximum normal stresses at the footing centre-line are calculated to be just 2\% - 4\% higher than the plasticity solution, as illustrated in Figure 6.13.
Theoretical loading

\[(\pi + 2) s_u + q\]

Computed loading

\[\sigma > (\pi + 2) s_u + q\]

Figure 6.13: Simplified Distribution of Normal Stresses on Clay Surface

It is suggested that, because of the nature of the normal stress distribution, the additional loading at the edges of the load spread area \((2B')\) allows the peak normal stress at the footing centre-line to slightly exceed the ultimate bearing capacity (Houlsby (1992)).

The total normal force (i.e. the area under the normal stress curve) applied at the unreinforced interface in the large strain analysis is approximately 13% greater than that applied in the small strain analysis. This is mostly due to the extra amount of footing load needed to overcome the surcharge that is created by the fill heave at the footing edge, although the particular distribution of shear stresses along the clay surface also has some influence, which is discussed later.

Figure 6.12 shows similar distributions of normal stress along the soil interfaces for a small and large strain analysis, but for the case where reinforcement is included. Considering the small strain analysis first, the normal stresses along the upper and lower surfaces of the reinforcement coincide, because there are no geometric effects involved. The increase in the magnitude of the small strain force applied at the interface for the
reinforced case compared to the unreinforced case, is approximately 16%. Which is entirely due to the fact that greater footing loads need to be applied for the stiffer reinforced case in order to obtain the same footing displacement.

The small strain, reinforced, normal stress distribution is noticeably different to the unreinforced distribution in that $\sigma$ equals the larger value of $5.71 \, s_u^{PS}$ at the footing centre-line and remains relatively constant, increasing only slightly, between $x = 0 \rightarrow 0.9 \, B$. The stress then decreases sharply to the permanent in-situ value of the fill self-weight at a point just beyond the calculated 95% load spread limit $\beta_{B40R}$.

Clearly, the horizontal extent of the area influenced by the footing pressure is slightly greater in the small strain reinforced case (B40RS) than for the small strain unreinforced case (B40S), i.e. $\beta_{B40RS} > \beta_{B40S}$. This would suggest that a small displacement reinforcement mechanism, of the form proposed by Giroud et al. (1984) involving an improvement of load distribution due to the inclusion of reinforcement, is operating in this two-layer soil system. Unfortunately the Giroud et al. (1984) design method only estimates the overall improvement due to the combined effects of subgrade confinement, load distribution and tensioned membrane (see Chapter 2), therefore no detailed predictions are given of how much improvement can be obtained in just the load spread angle. However, the increase in load spread for these particular small strain finite element calculations is only 5% for $s_u^{PS} = 30 \, kPa$, unlike the 30% average improvement reported by Alenowicz and Dembicki (1990) for $s_u^{PS} = 7.4 \, kPa$.

The profiles of the large strain reinforced normal stresses (run B40R) are prominently different for the upper and lower surfaces (Figure 6.12). They start at the footing centre-line almost identical at 6 $s_u^{PS}$ (which is about 12% greater than the ultimate vertical load capacity $P_u = 5.33 \, s_u^{PS}$ given by equation (6.3), because of the inward acting shear stresses on the clay, Figures 6.15 and 6.16 Mode 2), but then differ in that the lower stress distribution remains constant up to $x = 1.2 \, B$ and then drops sharply towards the load spread limit, which approximates to the plasticity theory solution of a constant load distribution. Whereas the upper large strain normal stress increases to a peak value of 8.5 $s_u^{PS}$ at the point where the curvature of the deformed membrane is greatest, before decreasing to the residual normal.

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stress. The normal force applied to the clay surface in the reinforced case (B40R) is 19% greater than that in the unreinforced (B40), but moreover the force on the upper surface applied by the fill base is 29% greater than in the unreinforced case. This larger amount of normal pressure sustained by the upper reinforcement surface compared to the lower surface is a consequence of the tensioned membrane action which develops at significant rut depths.

The 14% increase in the load spread for the large strain runs (between \( \beta_{B40} \) and \( \beta_{B40R} \)) is more substantial than the 5% achieved for the small strain runs (Table 6.1). However, these improvements are considered to be circumstantial since they are inconsistent with the changes in \( \beta \) for the parametric analyses listed in Tables 7.2, 8.1 and 10.1, where \( \beta \) is uninfluenced by the reinforcement.

Clearly a study of the normal stresses at the soil-reinforcement interfaces is instructive in investigating the mechanisms of reinforcement in greater detail, but likewise the shear stresses at the base of the fill and on the subgrade surface also have an important influence on the load bearing capacity of the two-layer soil system, i.e. the shear stress mechanism discussed in Chapter 2. Similar to Figures 6.11 and 6.12, the distribution of the small and large strain shear stresses (\( \tau \)) across the reinforcement length (\( x \)), at a final footing displacement of \( \delta = 0.6 \, B \), are shown in Figures 6.14 and 6.15 for the unreinforced and reinforced cases, respectively. Shear stresses acting outwards from the footing centre-line are positive.

In the unreinforced analyses, Figure 6.14, the upper and lower shear stress profiles superimpose for each finite element run. For the small strain calculation (B40S) the shear stresses are negligible directly under the footing, but reach a maximum that is equivalent to the clay shear strength (\( 1 \, s_{u}^{PS} \)) at about \( x = 1.6 \, B \). This peak of outward shear signifies some sliding between the clay and fill, localised at this point. The large strain calculation (B40) has more significant inward acting shear stresses in the area directly under the footing (\( \tau = -0.8 \, s_{u}^{PS} \)), but these change direction beyond \( x = 1.2 \, B \) and, as with the small strain analysis, an outward maximum of \( 1 \, s_{u}^{PS} \) is reached at \( x = 1.9 \, B \), which is the exact point of the load spread limit \( \beta_{B40} \).
Figure 6.14: Shear Stresses acting at the Unreinforced Soil Interface (at $\delta = 0.6B$)

Figure 6.15: Shear Stresses acting on the Reinforcement Surfaces (at $\delta = 0.6B$)

Outward acting shear stresses have the detrimental effect of reducing the bearing capacity of the subgrade (shown by Bolton (1979) and illustrated in Figure 6.16 as Modes 3 and 4), so that failure will occur at less than the ultimate vertical load capacity of $P_u = (\pi + 2)s_u^{ps} + q$. The presence of these outward shear stresses, therefore, has the effect of progressively decreasing the magnitude of the normal stresses away from the footing centre-line, as is apparent in Figure 6.11. However, inward acting shear stresses directly under the footing, as are present for the large strain analysis, will moderately enhance the
clay strength locally, which is also partly why the large strain normal stresses are slightly greater than the small strain ones in Figure 6.11 (although the soil heave at the fill surface is thought to have the greater influence).

In the reinforced analyses, Figure 6.15, the footing load has to be larger to attain the same footing displacement and consequently the shear stresses are also higher. Note that the shear stresses on the upper reinforcement surface are significantly larger than those in the unreinforced case. Inward shears of around \(-1.5 \sigma_u^{PS}\) develop directly under the footing and outward shears reach a peak of \(\tau = 3.6 \sigma_u^{PS}\) for small strain (B40RS) and \(\tau = 2.4 \sigma_u^{PS}\) for large strain (B40R). The small strain lower shear stresses vary between zero initially, to outward acting (with a maximum of \(0.7 \sigma_u^{PS}\)) and then to low inward acting, while the large strain lower shear stresses remain inward acting throughout (maximum of \(\tau = -0.6 \sigma_u^{PS}\)). It is this consistent inward shear stress (Mode 2 in Figure 6.16), which generates the higher load bearing capacity of approximately \(1.11 \times 5.33 \sigma_u^{PS} = 6 \sigma_u^{PS}\) seen in the large strain normal stress plot, Figure 6.12.

The small and large strain shear stresses on the underside of the reinforcement are relatively minor in comparison to both those shears acting on the upper reinforcement surface and those acting on the clay surface for the unreinforced case. This clearly illustrates the shear stress mechanism, where the reinforcement actually reduces the amount of outward clay shear stress and therefore increases its load bearing capacity. Furthermore, the reinforcement has the beneficial effect of increasing the amount of inward acting shear stress, especially for the large strain analysis, which will significantly enhance the strength of the clay. Using the theory of slip-line fields (Calladine (1985)) it is found that an 11% improvement in the ultimate bearing capacity of the clay can be obtained if the inward acting shear stresses, outside of the load spread area, reach the value of the clay shear strength, i.e. if \(\tau = \sigma_u^{PS}\) for Mode 2 in Figure 6.16.

These finite element investigations of the interface stresses reveal the reinforcement mechanisms operating in a two-layer soil system extremely effectively. It is apparent from these investigations, however, that the actual distribution and arrangement of the normal
and shear stresses in the unreinforced and reinforced situation, for small and large strain analyses, is much more complex than the simplistic assumptions made in the Houlsby et al. (1989) design method.

The distributions of the large and small strain reinforcement forces ($F_r$) are extracted from the finite element calculations by averaging the three Gauss point forces, per plane strain membrane element, and then plotting that average elemental force at the mean x co-ordinate of the Gauss points, as shown in Figure 6.17.

The tension induced in the reinforcement is due to the horizontal forces applied to it by the fill layer above and the clay subgrade below (as described in Burd and Brocklehurst (1990) and Kwok (1987)). Clearly there is a larger force developed throughout the reinforcement for a large strain analysis (B40R) compared to that for a small strain analysis.
Figure 6.17: Reinforcement Tension (at $\delta = 0.6B$)

(B40RS). At the centre-line the large strain reinforcement tension is 14% greater than for the small strain and the large strain peak tension is 19% greater than the small strain peak. Large displacement analyses evidently generate higher reinforcement forces due to the imposed deformation and the increased amount of force applied.

The Houlsby et al. (1989) analytical predictions of the small (RB40RS) and large (RB40R) displacement reinforcement tensions are shown in Figure 6.17. The reason the limit equilibrium design method underestimates the peak reinforcement forces is because of the inaccurate arrangement of stresses assumed in the analytical method. For instance, there is no consideration of the shear force applied to the reinforcement by the clay subgrade, which is in fact comparatively significant as illustrated in Figure 6.15. Furthermore, the analytically predicted value of $F_r$ is sensitive to the magnitude of the footing friction force ($P_f B \tan \delta_r$), which for RB40RS is lower than for RB40R, because of the smaller angle of friction at the footing base (Table 6.1).

6.3.6 Conclusions from Preliminary Experiments

The preliminary plane strain finite element studies, discussed in this Section 6.3, produced several important conclusions relevant to the selection of the most appropriate numerical
approach for the parametric study, for example;

Refinement of a finite element mesh (i.e. increasing the number of elements and therefore nodes) leads to a greater accuracy in the numerical calculation, but at the expense of computing time. A balance has to be found so that the analysis is relatively efficient and practical, but with an acceptably small inherent error. The typical plane strain mesh that was finally chosen for the parametric study, after the completion of a convergence study (Section 6.3.2), is shown in Figure 5.2. The ratio of the width of the mesh to the element side length, of the smallest element employed in the mesh, is 350 and for the largest element is 3.

Using too small a mesh spuriously increases the elastic stiffness of the load-displacement response, but then decreases the plastic stiffness, so that the predicted final load bearing capacity of the system is considerably reduced (by $= 20\%$). The interface stresses and reinforcement forces are also influenced by the position of the mesh boundaries, particularly the normal stresses which are significantly reduced if the boundaries are too close to the footing. The minimum mesh width capable of negating the boundary effects is believed to be about $10B$ and the minimum overall depth is approximately $13.2B$ (including 1.2 $B$ depth of fill).

A large displacement, large strain type analysis is essential in order to capture fully all the reinforcement mechanisms acting in a two-layer soil system. However, the shear stress mechanism of the reinforcement, which reduces both the shear stress magnitude and the area of the clay surface it influences, operates even if there is no surface rutting or membrane deformation. This small displacement mechanism, combined with a slightly increased load spread, provides the greatest amount of improvement in the load bearing capacity, accounting for about 25% strengthening in the particular case considered. In a large strain analysis, where significant membrane deformations can occur, these small displacement mechanisms have an increased influence and the additional large displacement tensioned membrane and restraint mechanisms also develop. These geometrically non-linear effects occur as a result of substantial rutting and contribute an extra 5% to the overall loading improvement obtained for this specific case.
Therefore, the design methods for two-layer soil systems, such as unpaved roads, need to take account of both the small displacement reinforcement mechanisms and those mechanisms which, it has been shown, also act at large displacements. The Houltsby et al. (1989) method is shown to provide good approximations to both the small displacement unreinforced and reinforced limit loads. It is also capable of accurate predictions to the large displacement reinforced load, although less exact in predicting the large displacement unreinforced load and the reinforcement forces.

A method is proposed of estimating the load spread angle, $\beta$, from the upper interface normal stress distributions. This provides a consistent parameter which can be compared in each of the finite element studies to assess the influence of a particular variable.

6.4 Comparison of Plane Strain and Axisymmetric Analyses

A comparison of a plane strain and an axisymmetric large strain analysis is done for the 'central parametric analysis' using the material properties and dimensions defined in Appendix 6A, for the unreinforced and reinforced cases.

The plane strain finite element mesh with six-noded continuum elements, three-noded reinforcement elements and six-noded interface elements is shown in Figure 5.2 and the corresponding axisymmetric mesh with fifteen-noded continuum elements, the new five-noded reinforcement elements and ten-noded interface elements is shown in Figure 6.18. To obtain an accurate comparison between the finite element results the overall dimensions of the two meshes are identical and a similar number of nodes are used in each analysis. The plane strain mesh contains 1036 nodes and the axisymmetric mesh contains 1197 nodes. The footing radius, $R$, for the axisymmetric analysis is equal to the footing half width, $B$, for the plane strain analysis, i.e. $R = B = 0.25\ m$. 

6-33
Figure 6.18: Axisymmetric Finite Element Mesh AXB

6.4.1 Load-Displacement Response

The footing pressure-displacement responses for each of the four large strain finite element runs are illustrated in Figure 6.19 and the results are presented in Table 6.2, where $P$ is the average value of vertical stress beneath the footing and $\delta$ is the vertical footing displacement.

Using the load spread angles, $\beta$ (as determined by the 95% load spread limit method), and the angles of friction of the footing base, $\delta_f$ (calculated by the procedure described in Section 6.5.7), each given in Table 6.2, a direct comparison can be made between the calculated finite element results and the results predicted by the analytical methods proposed by Houlsby et al. (1989) for plane strain conditions and Houlsby and Jewell (1990) for axisymmetric conditions (see Chapter 2). The ultimate footing pressures, $P_f$, given by the two analytical methods are shown in Figure 6.19 and tabulated in Table 6.2, where the ‘R’ denotes a reinforced analysis and ‘U’ an unreinforced analysis.
**Figure 6.19: Pressure-Displacement Responses for Plane Strain and Axisymmetric Analyses**

<table>
<thead>
<tr>
<th></th>
<th>F.E.</th>
<th>Analytical</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\delta_f$</td>
<td>$\beta$</td>
</tr>
<tr>
<td><strong>Axi</strong></td>
<td>R</td>
<td>$-6.5^\circ$</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>$-10.5^\circ$</td>
</tr>
<tr>
<td><strong>P.S.</strong></td>
<td>R</td>
<td>$-10.3^\circ$</td>
</tr>
<tr>
<td></td>
<td>U</td>
<td>$-8.0^\circ$</td>
</tr>
</tbody>
</table>

**Table 6.2: Run References for the Finite Element and Analytical Results Shown in Figure 6.19**

The initial finite element pressure-displacement responses for the axisymmetric analyses are a little more than twice as stiff as the responses for the plane strain analyses. Furthermore, the responses for the axisymmetric unreinforced and reinforced analyses diverge at about $\delta = 0.05 \, R$, whereas the corresponding plane strain analyses do not separate until about $\delta = 0.09 \, B$. 

6-35
At the final vertical footing displacement of $\delta = 0.6 R$; $B$ the improvement in the load capacity of the system by including reinforcement in the axisymmetric analysis is about 57%, while the improvement for the plane strain analysis is only 30%. Additionally, the stiffer response of the reinforced axisymmetric analysis beyond $\delta = 0.2 R$; $B$, compared to the reinforced plane strain analysis, is due to the increased amount of tensioned membrane effect. This shows that the reinforcement has a greater effect in improving the allowable loading on the two-layer soil system under axisymmetric conditions compared to plane strain conditions.

The ultimate footing loads predicted by the Houlbsby et al. (1989) and Houlbsby and Jewell (1990) analytical methods, Table 6.2, tend to underestimate the finite element results, particularly in the axisymmetric case. This is because of the limitation of the analytical methods to small footing displacements, whereas large displacements are allowed to develop in the large strain finite element solutions. Moreover, these methods are applicable to unpaved road design and if an unpaved road becomes rutted, then it is clear that the geometry of the system is such that the loading cannot be treated as axisymmetric.

The analytically predicted improvements in $P_f$ from including reinforcement are overestimated, especially in the axisymmetric case where the improvement is estimated at approximately 91%, compared to the 57% improvement in the finite element solutions. This is because the unreinforced limit loads are predicted too low, due to the erroneous assumption that there are constant outward acting shear stresses at the unreinforced clay surface and that these are increased by the outward frictional forces on the footing base, therefore further decreasing $N_c$, whereas the shear stresses directly under the footing are actually found to be inward acting, Figure 6.20.

No significant changes occur in the load spread angles, given in Table 6.2, between the reinforced and unreinforced analyses, but there is a general decrease in $\beta$ between the plane strain and axisymmetric cases. This is because the axisymmetric load spread gives a more rapid reduction of vertical stress within the fill than occurs in plane strain, as concluded by Milligan et al. (1989).
6.4.2 Normal and Shear Stress Distributions

The normal ($\sigma$) and shear ($\tau$) stress distributions along the upper and lower reinforcement surfaces, at the final footing displacement of $\delta = 0.6 \, R$, are extracted from the appropriate interface elements and shown in Figure 6.20 normalised with respect to the plane strain value of $s_\text{w}^{ps}$. For all the plane strain analyses the three Gauss point stresses for each interface element are averaged and plotted at the mean horizontal distance from the footing centre-line in order to obviate spurious oscillations, whereas for the axisymmetric analyses the seven Gauss point stresses for each interface element are plotted individually.

The differences between the axisymmetric and plane strain stress distributions plotted in Figure 6.20 are, on the whole, small. For the unreinforced case the axisymmetric normal stresses, which are slightly greater than the plane strain normal stresses directly under the footing, start at $\sigma = 5.48 \, s_\text{w}^{ps}$ on the centre-line and increase to a maximum of $\sigma = 7.05 \, s_\text{w}^{ps}$ within a small horizontal distance, beyond which the stresses decrease steadily to the constant surcharge of the fill self weight ($q = \gamma_{\text{f}} D_f$).

In Section 6.3.5 the magnitude of the plane strain normal stresses on the clay surface was compared to the ultimate bearing capacity ($P_u$) predicted by plasticity theory, equation (6.3). For the case of purely vertical loading in plane strain conditions the bearing capacity factor, $N_c$, is exactly equal to $(\pi + 2)$ and if outward shear stresses act on the clay surface the reduction in $N_c$ can be expressed analytically. However, for the axisymmetric case the calculation of $N_c$ cannot be done analytically, but it may be estimated by the method of characteristics. Houlbysy and Jewell (1990) give a table of combinations of $N_c$ and the average outward shear stress calculated using the "Fields" program developed by Professor G. T. Houlbysy at the University of Oxford. The full bearing capacity, for purely vertical loading, is found to be 5.69.

Unlike the plane strain case the axisymmetric normal stresses illustrated in Figure 6.20 cannot be related to the plasticity calculations of $P_u$ using $N_c = 5.69$, because this represents the collapse load for a Tresca material and the clay is represented in the finite element solution as an elastic-perfectly plastic von Mises material (Chapter 3), for which there is no known exact collapse load. However Yu (1990) shows that the von Mises collapse load is

6-37
Figure 6.20: Normal Stress ($σ/s_u^{PS}$) and Shear Stress ($τ/s_u^{PS}$) along Interface for Plane Strain and Axisymmetric Analyses, (at $δ = 0.6 \; B; \; R$)

greater than the solution for a Tresca material and less than the solution for the Tresca material multiplied by a factor of $2/\sqrt{3}$. This is because in triaxial stress states the von Mises yield surface is equivalent to the Tresca yield surface and in plane strain conditions the von Mises yield surface represents the Tresca surface scaled by a factor of $2/\sqrt{3}$. The
stress state created by the circular load spread area at the fill base (synonymous to a loaded circular smooth footing) corresponds neither to a triaxial stress state nor plane strain conditions. The actual collapse load for a circular smooth footing resting on a von Mises soil is shown by Yu (1990) to be approximately \( N_c = 6.3 \). Therefore the normal stress on the clay surface within the load spread area for the axisymmetric finite element runs, when no shear stresses act, can be expected to be a constant value of:

\[
P_u = N_c s_u^T + q = 6.3 \frac{\sqrt{3}}{2} s_u^{PS} + q
\]  

(6.4)

Clearly the axisymmetric normal stresses at the unreinforced fill-clay boundary do not give this theoretical uniform distribution of \( \sigma = 5.46 s_u^{PS} + q = 5.46 s_u^{PS} + 0.19 s_u^{PS} = 5.65 s_u^{PS} \) for the particular value of \( \gamma_f D_f \), which is because of the outward acting shear stresses beyond \( r = 1.2 R \) shown in Figure 6.20. However, the normal stress at the footing centre-line is only 3% smaller than the plasticity solution, which is a good agreement. Similarly, the plane strain normal stress at the footing centre-line is just 4% greater than the plasticity solution given by equation (6.3), as discussed in Section 6.3.5.

The unreinforced axisymmetric and plane strain shear stresses initially act inwards, reaching a maximum of about \( \tau = -0.8 s_u^{PS} \), and then reverse direction to become significant outward shear stresses of \( \tau = 1 s_u^{PS} \), which cause the reduction in the normal stresses. Beyond about \( r = 3 R \) and \( x = 3 B \) the shear stresses remain as very small inward acting stresses.

The reinforced upper normal stresses are rather larger for the axisymmetric case than for the plane strain case, within the area directly under the footing, illustrating the greater tensioned membrane effect, see Section 6.4.1. However the reinforced lower normal stresses for the axisymmetric and plane strain cases are less disparate. The lower normal stresses for the axisymmetric case show a few spurious oscillations, the slightly smaller plane strain normal stresses showed some similar oscillations before a procedure of averaging Gauss point stresses was used. The expected constant value of \( \sigma \) within the load spread area, given by plasticity theory, is \( \approx 5.65 s_u^{PS} \) for axisymmetry and \( \approx 5.33 s_u^{PS} \) for plane strain. In comparison the axisymmetric finite element solution gives \( \sigma_{uu} = 5.68 s_u^{PS} \) at the footing centre-line (which is 0.5% larger than the theoretical prediction) and an average of 6.26 \( s_u^{PS} \)
between \( r = 0 \rightarrow 1.2 R \), which is 11\% larger than the plasticity answer, while the plane strain finite element solution gives a consistent \( \sigma_{ps} = 6 s_u^{ps} \) between \( x = 0 \rightarrow 1.2 B \), which is approximately 12\% larger. These increased normal stresses are due to the inward acting shear stresses on the clay outside of the load spread area (Figures 6.20 and 6.16 Mode 2). For the limiting case where \( \tau = -1 s_u^{ps} \) the exact increase in \( N_c \) can be estimated analytically in the plane strain case, as given in Figure 6.16, whereas in axisymmetry this is not possible, although it is likely that the increase is of approximately the same magnitude.

The reinforced axisymmetric and plane strain upper shear stresses are very alike and follow a similar distribution pattern to that of the unreinforced shear stresses, but reach much higher magnitudes because of the larger \( P_f \) needed to attain the same \( \delta \). The outward acting axisymmetric upper shear stresses display a more erratic distribution with a spuriously fluctuating peak value, compared to the plane strain, but the overall trends are very similar.

The reinforced axisymmetric and plane strain lower shear stresses are again quite comparable. Between \( r = 0 \rightarrow 0.8 R \) and \( x = 0 \rightarrow 0.8 B \) the shear stresses are zero for the plane strain case and vary from very small inward to very small outward for the axisymmetric case, respectively. Beyond this point the shear stresses maintain a continually inward acting distribution, with a peak of \( -1 s_u^{ps} \) for the axisymmetric analysis and \( -0.6 s_u^{ps} \) for the plane strain analysis.

### 6.4.3 Reinforcement Tension Distribution

The longitudinal reinforcement tension, \( F_r \), in the plane strain analysis and the tangential and circumferential tensions in the axisymmetric analysis, at the final footing displacement of \( \delta = 0.6 R \); \( B \) are plotted in Figure 6.21. The value of \( F_r \) in the plane strain case is calculated as the average of the three Gauss point forces per plane strain membrane element, whereas each axisymmetric membrane element Gauss point is plotted individually.

The axisymmetric tangential and circumferential membrane tensions show a similar pattern of distribution, but with the tangential reaching a peak of about \( F_r = 4 s_u^{ps} \) and the circumferential a peak of only \( F_r = 3.4 s_u^{ps} \). Beyond \( r = 2.2 R \) the tangential membrane tension reduces to zero quickly, in terms of horizontal distance, while the circumferential
tension decreases to zero over a much larger distance. By comparison the plane strain longitudinal membrane tension reaches a maximum approximately equal to the peak axisymmetric tangential tension, but decreases more gradually than either of the axisymmetric tensions. This is because the load spread area in the plane strain analysis is larger than in the axisymmetric analysis and consequently the horizontal forces applied to the membrane by the upper and lower shear stresses extends over a greater horizontal distance.

Both the Houlsby et al. (1989) prediction of the reinforcement force for the plane strain case (RB40R) and the Houlsby and Jewell (1990) prediction for the axisymmetric case (RAXB40R) underestimate the peak reinforcement forces from the equivalent finite element solutions, as shown in Figure 6.21. This is because of the inexact simplified stress distribution assumed in these analytical methods.

6.5 Results of the Plane Strain and Axisymmetric Central Analyses

The final results from the unreinforced and reinforced plane strain and axisymmetric central parametric analyses (B40, B40R, AXB40 and AXB40R as defined in Appendix 6A) are presented fully in this section and a detailed investigation of the finite element runs is
undertaken. The 'plates', referred to in the proceeding sections and presented in Appendix 6B, were obtained from the two-dimensional contouring program '2Can' written by Professor G. T. Houlby at the University of Oxford.

The areas of investigation are; 1) the deformations within the meshes directly under the footing at the end of the analyses, 2) the actual nodal displacements within the area of greatest deformity, 3) the horizontal strains under the footing, 4) the normal and shear stress distributions under the footing, 5) the principal stress directions in the fill, 6) the mobilised angle of friction for the fill, and 7) the stress distributions on the underside of the footing and the corresponding angle of friction of the footing base.

6.5.1 Deformed Meshes and Interface slip

The final deformed plane strain and axisymmetric meshes, within the footing area, for the unreinforced and reinforced analyses are shown in Figures 6.22 a) and b).

Clearly there is a reduced amount of deformation and heave in the fill layer for the reinforced analyses, compared to the unreinforced ones. Note the one single overturned continuum element at the fill surface, just at the outside edge of the footing in both the plane strain and axisymmetric deformed meshes, which has occurred because of excessive displacement. This does not significantly effect the accuracy of the results, but could be avoided by using an even higher density of elements within this area of large deformation.

The lines of displaced interface and/or membrane elements are illustrated in Figure 6.22 by a slightly thicker line. Directly under the footing there is no visible evidence of slip between the upper and lower surfaces of the boundary in the unreinforced or the reinforced analyses, but at the point of reverse curvature there is a small amount of observable slip in both analyses. This is reflected in Figure 6.23, which shows the relative movement \( (u_r) \) at the unreinforced and reinforced soil interfaces for the plane strain and axisymmetric analyses. Positive \( u_r \) indicates movement of the upper surface to the right relative to the lower surface. In axisymmetry slipping occurs within \( r = 4 \, R \) of the footing centre-line, whereas for plane strain slip continues to occur up to \( x = 6 \, B \). The slip at the unreinforced and upper reinforced interfaces is initially negative with the clay moving out from under the fill, generating the
inward shear stresses shown in Figure 6.20, but then reverses at $x; r = 1.5 \, B; R$, which develops the outward shear. The lower reinforced interface sustains consistently negative slip and therefore inward shear, i.e. the shear stress mechanism of the reinforcement. Beyond about $3 \, B; R$ the slip does not generate any significant shear because of the small amount of normal stress acting.
6.5.2 Nodal Displacements

The nodal displacement vectors within the area of high deformation are illustrated without magnification in Figures 6.24 a) and b), Appendix 6B, for the plane strain and axisymmetric analyses. The initial positions of the footing and the boundary between the fill and clay are shown.
On comparing the unreinforced and reinforced nodal displacement vector plots in Figure 6.24 it is clear that when reinforcement is included both the magnitude of the displacements is greatly reduced and the mechanism of deformation alters for both plane strain and axisymmetry. For the unreinforced analyses the mode of deformation appears to be similar to that illustrated in both Figure 6.2 and Figure 6.16 Mode 1, where the change in the material type has little effect. Incorporating the reinforcement, however, has the effect of developing separate deformation mechanisms in each material, where the fill has a confined deformation zone just outside of the load spread, while the failure within the clay extends deeper as illustrated by the areas of yield in Figures 6.5 and 6.26. Immediately beneath the footing, for the unreinforced and reinforced cases, the fill is subjected to a rigid wedge type displacement. Although in plane strain and axisymmetry the changes in the deformation characteristics between unreinforced and reinforced are similar, the magnitude of the displacements reduces more quickly, in terms of distance, for axisymmetry.

6.5.3 Horizontal Strains in the Fill

The horizontal strains in the fill and clay layers for the plane strain and axisymmetric analyses are shown in Plates 6.1 and 6.2 Appendix 6B, respectively, for the unreinforced and reinforced cases at $\delta = 0.6B; R$. The strains are given by the final Gauss point strains, but are plotted at the original Gauss point co-ordinates with the initial positions of the footing and the fill-clay boundary indicated.

The strains are found to be extremely large in a small area directly beneath the footing edge ($\approx 100\%$) and are moderately greater in the axisymmetric analyses compared to the plane strain. The reinforcing effect of reduced horizontal strains at the fill base is clearly seen by comparing the unreinforced and reinforced contour plots, i.e. the restraint mechanism.

6.5.4 Vertical and Shear Stresses in the Fill

Plates 6.3 and 6.4, Appendix 6B, depict contour plots of the vertical stresses through the fill layer normalised by the plane strain clay shear strength ($\kappa_a^{ps} = 30 kPa$) at $\delta = 0.6B; R$ for
the plane strain and axisymmetric analyses respectively. Similarly, Plates 6.5 and 6.6 show
the distributions of shear stress for the plane strain and axisymmetric analyses. The stresses
are given by the final Gauss point stresses and plotted at the original Gauss point co-ordinates.

The vertical stress plots clearly show the load spread beneath the footing and the
concentration of higher vertical stresses to the outside edge of that load spread. There is
also a noticeable reduction in the vertical stress across the reinforcement. The shear stresses
at the fill base for the unreinforced and reinforced analyses are inward acting at the centre-line
and become outward acting towards the edge of the vertical load spread, whereas, at the top
of the fill directly underneath the footing, the shear stresses are purely outward acting (see
Section 6.5.7). For the reinforced case the magnitude of the shear stresses is significantly
larger, due to the membrane restraining the fill from moving outwards, and the arching effect
described in Section 6.5.5 is visible. The magnitude of both the vertical and shear stresses
are slightly higher for axisymmetry compared to plane strain.

The stresses along the fill base are the same as those given by the interface elements
in Figures 6.20, although any minor discrepancy is due to the fact that the continuum element
Gauss points are slightly removed from the interface element Gauss points.

6.5.5 Principal Stress Directions in the Fill

From the final horizontal, vertical and shear stresses at each Gauss point (at \( \delta = 0.6 \ B \); \( R \))
the principal stress directions in the fill layer are plotted in Figures 6.25 a) and b), Appendix
6B, at the final Gauss point co-ordinates for the plane strain and axisymmetric analyses
respectively.

In the unreinforced analyses the principal stress directions in the fill layer are
predominantly vertical directly under the footing and horizontal outside of the load spread.
For the reinforced analyses, however, the fill layer appears to be subjected to an arching
effect beneath the footing where the upper half of the fill layer is subjected to higher
horizontal than vertical stresses, whereas the lower half of the fill sustains predominantly
vertical stresses. A similar effect was detected in some of the embankments analysed by
6.5.6 Mobilised Friction Angle and Yield Area

The mobilised friction angle, $\phi'$, of the fill can be calculated from the actual stresses acting at each Gauss point by equation (6.5):

$$\sin \phi' = \frac{(\sigma_x^2 - 2 \sigma_x \sigma_y + \sigma_y^2 + 4 \tau_{xy}^2)^{\frac{1}{2}}}{(\sigma_x + \sigma_y)}$$

(6.5)

Plate 6.7 a) shows the calculated absolute values of $\phi'$ throughout the fill for the plane strain unreinforced analysis at a footing displacement of $\delta = 0.6 \ B$, while Plates 6.7 b) and c) show $\phi'$ for the plane strain reinforced analysis at $\delta = 0.2 \ B$ and $\delta = 0.6 \ B$ respectively. Similarly, the axisymmetric unreinforced and reinforced analyses at $\delta = 0.6 \ R$ only, are illustrated in Plates 6.8 a) and b) respectively. The vertical dimension in Plates 6.7 and 6.8 is scaled by a factor of 5 for the sake of clarity.

The fill material possesses a peak plane strain friction angle of $\phi_{ps} = 40^\circ$ and from Plate 6.7 b) it is clear that the majority of the material in plane strain conditions has reached a state of yield at a footing displacement of only $\delta = 0.2 \ B$, which is why Figure 6.5 shows the entire fill layer to have yielded. In contrast the area of yield for the axisymmetric analyses, Figure 6.26, is much smaller and less of the fill material has yielded.

![Unreinforced](image1)

![Reinforced](image2)

Figure 6.26: Yielded Gauss Points for the Axisymmetric Runs at $\delta = 0.6 \ R$
By comparing the principal stress directions in Figure 6.25 a) and $\phi'$ in Plates 6.7 it is clear that for plane strain conditions all of the fill is in a state of passive failure outside of the load spread area for the unreinforced and reinforced cases. Whereas, beneath the footing $\phi'$ is less than $\phi$ of the soil and therefore the horizontal stress is greater than active conditions, particularly for the unreinforced case. This higher horizontal stress increases the tension in the reinforcement, which is partly why Houlsby et al. (1989) underestimates $F_r$ (Figure 6.21). For axisymmetric conditions however, Figure 6.25 b) and Plates 6.8, a large proportion of the fill material has not yielded, particularly in the areas furthest from and directly beneath the footing. This demonstrates that the assumption in Houlsby et al. (1989) and Houlsby and Jewell (1990) of passive failure conditions immediately beyond the footing is correct, but that the assumed active failure conditions under the footing is inaccurate.

6.5.7 Stress Distributions on the Underside of the Footing

The distributions of the normal ($\sigma$) and shear ($\tau$) stresses on the underside of the footing, for the unreinforced and reinforced analyses, are extracted from the continuum element Gauss points using Method 4, as described in Section 5.2 (two nearest Gauss points for plane strain and five nearest for axisymmetry), and plotted in Figures 6.27 a) and b) for the plane strain and axisymmetric analyses respectively.

The shear stresses shown in Figures 6.27 a) and b) are mostly outward acting stresses, which contradict the assumption made in Houlsby et al. (1989) and Houlsby and Jewell (1990) of inward acting footing shear forces. The small inward acting shear stresses at the footing edge, in the plane strain and axisymmetric analyses, are likely to be the results of discretization errors.

The integral of the Gauss point stress distributions gives the shear and normal forces per unit area acting at the footing-fill boundary. This normal force is consistent with the final nodal footing pressures, $P_f$ for the runs shown in Figure 6.19. Clearly the normal and shear forces are larger for the reinforced analyses than for the corresponding unreinforced and furthermore in axisymmetry the normal forces are greater than in plane strain.
Figure 6.27: Normal and Shear Stresses along underside of Footing at $\delta = 0.6 \frac{B}{R}$

An angle of friction at the footing base, $\delta_f$, can be established from equation (6.6):

$$\delta_f = \tan^{-1} \frac{\tau_f}{P_f}$$

(6.6)

where $\tau_f$ is the total shear force per unit area at the footing base and $P_f$ is the total normal footing pressure.

A value of $\delta_f$ is calculated for each finite element run and used in the relevant analytical computations, maintaining the convention that a negative sign indicates an outward acting footing frictional force which is consistent with the limit equilibrium methods.
APPENDIX 6A

Material Properties of the Central Parametric Analyses

WHEEL or FOOTING

Perfectly Rough Contact

- $B$ Footing Half Width for Plane Strain $0.25 \, m$
- $R$ Footing Radius for Axisymmetry $0.25 \, m$
- $\delta$ Rut Depth $0.15 \, m$

FILL SUBBASE

- $G_f$ Shear Modulus $6000 \, kPa$
- $\nu_f$ Poisson’s Ratio $0.2$
- $\gamma_f$ Bulk Unit Weight $19 \, kN/m^3$
- $D_f$ Fill Depth $0.3 \, m$
- $\phi_T$ Triaxial Friction Angle $35.6^\circ$
- $\phi_{PS}$ Plane Strain Friction Angle $40^\circ$
- $\gamma_a$ Degree-of-Association $0.6$
- $\psi_{PS}$ Plane Strain Dilation Angle $24^\circ$

CLAY SUBGRADE

- $G_c$ Shear Modulus $1600 \, kPa$
- $\nu_c$ Poisson’s Ratio $0.49$
- $\gamma_c$ Bulk Unit Weight $19 \, kN/m^3$
- $D_c$ Clay Depth $5 \, m$
- $s_{u}^{T}$ Undrained Triaxial Shear Strength $26 \, kPa$
- $s_{u}^{PS}$ Undrained Plane Strain Shear Strength $30 \, kPa$

REINFORCEMENT MEMBRANE

- $J$ Reinforcement Stiffness $300 \, kN/m$
- $L$ Reinforcement Length $5 \, m$
- $\nu_r$ Poisson’s Ratio $0$
- $t_0$ Initial Thickness unity
- $K$ Reinforcement Bulk Modulus $K = J \frac{t_0}{(1-v_c^2)} = 300 \, kN/m$

INTERFACE PROPERTIES

- $k_s$ Shear Interface Stiffness $100 \, 000 \, kN/m$
- $k_n$ Normal Interface Stiffness $200 \, 000 \, kN/m$
- $\phi_i$ Friction Angle $35^\circ$
- $\psi_i$ Dilation Angle $10^\circ$
APPENDIX 6B

Contours and Trajectories of the Central Parametric Analyses

Plane Strain

Unreinforced

Reinforced

Figure 6.24 a): Nodal Displacement Vectors in Plane Strain
Figure 6.24 b): Nodal Displacement Vectors in Axisymmetry
a) Unreinforced and b) Reinforced

Plate 6.1: Horizontal Strains (Eps-X %) in Plane Strain at $\delta = 0.6B$
a) Unreinforced and b) Reinforced

Plate 6.2: Horizontal Strains (Eps-X %) in Axisymmetry at $\delta = 0.6 R$
a) Unreinforced and b) Reinforced

Plate 6.3: Vertical Stresses (\(\text{Sigma - Y/Su}\)) in Plane Strain at \(\delta = 0.6 \, B\)
a) Unreinforced and b) Reinforced

Plate 6.4: Vertical Stresses (\( \Sigma Y/Su \)) in Axisymmetry at \( \delta = 0.6 \, R \)
a) Unreinforced and b) Reinforced

Plate 6.5: Shear Stresses (Tau/Su) in Plane Strain at $\delta = 0.6 \, B$
Plate 6.6: Shear Stresses (\(\tau/\sigma\)) in Axisymmetry at \(\delta = 0.6R\)

a) Unreinforced and b) Reinforced
Figure 6.25 a): Plane Strain Principal Stress Directions in the Fill at $\delta = 0.6 \, B$
Figure 6.25 b): Axisymmetric Principal Stress Directions in the Fill at $\delta = 0.6 R$
Plate 6.7: Mobilised Angles of Friction ($\phi^*$) in Plane Strain

a) Unreinforced at $\delta = 0.6 \, B$ and b) Reinforced at $\delta = 0.2 \, B$
c) Reinforced at $\delta = 0.6B$

Plate 6.7: Mobilised Angles of Friction ($\phi'$) in Plane Strain
a) Unreinforced and b) Reinforced

Plate 6.8: Mobilised Angles of Friction ($\phi'$) in Axisymmetry at $\delta = 0.6\ R$
CHAPTER 7:

INFLUENCE OF FILL DEPTH AND
FRICITION ANGLE

7.1 Introduction

This chapter describes an investigation into the plane strain response of the two-layer soil system for variations in both fill depth and fill strength. Through careful selection of these important design parameters, the maximum benefit can be obtained from including reinforcement within a two-layer soil system.

The typical reinforced two-layer soil system, described as the 'central parametric analysis' in Chapter 6, is used as the datum for this investigation with the material properties given in Appendix 6A. Three separate fill depths and friction angles are used in this study for both the reinforced and unreinforced cases. Table 7.1 gives the varying values of the fill depth $D_f (m)$ and the plane strain fill friction angle $\phi_{ps}$ (degs) for the eighteen large strain finite element runs, (postscript 'R' designates reinforced). The different fill depths selected are representative of those thicknesses commonly used in practical and experimental unpaved roads and reinforced subsoil trials, as discussed in Chapter 2.

The friction angles used in this study cover the typical range of fill strengths that are encountered in two-layer soil systems. Milligan et al. (1989) proposes that, for an unpaved road, a fill strength of $\phi = 30^\circ$ is "weak", $\phi = 38^\circ$ is "normal" and $\phi = 45^\circ$ is "strong". Poran (1985) suggests that the range of friction angles of dense granular materials are normally between $35^\circ$ to $50^\circ$ and he uses a triaxial friction angle of $\phi = 46^\circ$ in the finite element model. Hird and Kwok (1990) use $\phi_{ps} = 40^\circ$ in their embankment analysis, and Alenowicz and
<table>
<thead>
<tr>
<th>Friction Angle $\phi_{ps}$</th>
<th>Fill Depth $D_f$</th>
<th>0.6 $B = 0.15 \text{ m}$</th>
<th>1.2 $B = 0.3 \text{ m}$</th>
<th>2.0 $B = 0.5 \text{ m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 °</td>
<td>A30</td>
<td>B30</td>
<td>C30</td>
<td></td>
</tr>
<tr>
<td>A30R</td>
<td></td>
<td>B30R</td>
<td>C30R</td>
<td></td>
</tr>
<tr>
<td>40 °</td>
<td>A40</td>
<td>B40</td>
<td>C40</td>
<td></td>
</tr>
<tr>
<td>A40R</td>
<td></td>
<td>B40R</td>
<td>C40R</td>
<td></td>
</tr>
<tr>
<td>55 °</td>
<td>A55</td>
<td>B55</td>
<td>C55</td>
<td></td>
</tr>
<tr>
<td>A55R</td>
<td></td>
<td>B55R</td>
<td>C55R</td>
<td></td>
</tr>
</tbody>
</table>

**Table 7.1: Finite Element Run References**

Dembicki (1990) use sand of $\phi = 30^\circ$ in their experimental model temporary road tests. Furthermore, as stated in Chapter 6, Type 1 Subbase Material is typically used in unpaved roads (Specification for Highway Works (1986)), which has an average peak friction angle of between $\phi_{peak} = 46^\circ$ and $56^\circ$.

### 7.2 Fill Friction Angle

There is a variety of different instruments and methods available to measure the friction angle of a granular material and because $\phi$ is sensitive to these different methods a variety of different friction angle values can be obtained for one particular material. To avoid confusion the specific classification test method used must be stated in conjunction with the actual parameter of friction angle. For the particular finite element calculations undertaken in this study the triaxial compression friction angle, $\phi_r$, is the required input parameter for the Matsuoka constitutive model used in OXFEM (see Chapter 3), and a conversion is then made so that the numerical calculations are actually for a specified plane strain friction angle.

The relationship between the triaxial and plane strain plasticity parameters of granular materials (i.e. $\phi_r$, $\gamma_s$ and $\phi_{ps}$, $\psi_{ps}$ respectively) is comprehensively discussed by Burd (1986), where a table of comparisons of $\phi_r$ and $\phi_{ps}$ is given for different values of dilation rate. The difference between triaxial and plane strain friction angles was also discussed by Wroth (1984), who proposed the linear empirical relationship:

$$8 \phi_{ps} = 9 \phi_r$$  \hspace{1cm} (7.1)
The dilation characteristics of the fill material are thought to have only a small influence on the load capacity behaviour of a soil under a footing, because the soil is relatively unrestrained kinematically compared to, say, the highly constrained soil near a pile (Houlsby (1991)). Therefore, the degree-of-association parameter ($\gamma_a$) is taken to be a constant value of 0.6 throughout this parametric study. An exhaustive review of how the dilatancy of soils affects their behaviour is made by Houlsby (1991).

7.3 Parametric Study

The following sections consider the effects of variations in both the fill depth and fill strength on the load-displacement response of the two-layer soil structure, the arrangement of stresses and displacements at the reinforced and unreinforced fill-clay interface and the reinforcement tension, as predicted by the finite element analysis. For comparison, the analytical results given by the Houlsby et al. (1989) method are also presented for the same variations in fill depth and strength. The effect on the load spread through the fill is also investigated, with comparisons made to previous experimental work and theoretical analyses.

7.3.1 Load-Displacement and Load Spread Responses

The load-displacement responses for each of the eighteen finite element runs listed in Table 7.1 are shown in Figure 7.1, where $P$ is the vertical footing pressure and $\delta$ is the vertical footing displacement.

The final footing pressures, $P_f$, at a footing displacement of $\delta = 0.6 B$ are listed in Table 7.2, along with the estimated load spread angles, $\beta$, (calculated as the 95% normal stress limit) and the angles of friction of the footing base, $\delta_f$, (determined by the method discussed in Section 6.5.7) for each analysis. Additionally, the ultimate footing loads and the reinforcement forces predicted by the Houlsby et al. (1989) analytical method are also given in Table 7.2, calculated using the relevant angles of $\beta$, $\delta_f$ and $\phi_{ps}$. 
Figure 7.1: Pressure-Displacement Responses for Different Fill Depths ($D_f$) and Friction Angles
<table>
<thead>
<tr>
<th>Ref.</th>
<th>( \frac{P_f}{s_u^{PS}} )</th>
<th>( \beta \degree )</th>
<th>( \delta_f \degree )</th>
<th>Analytical</th>
</tr>
</thead>
<tbody>
<tr>
<td>A30</td>
<td>7.23</td>
<td>37.2</td>
<td>-7.7</td>
<td>RA30</td>
</tr>
<tr>
<td>A40</td>
<td>7.64</td>
<td>37.4</td>
<td>-6.2</td>
<td>RA40</td>
</tr>
<tr>
<td>A55</td>
<td>7.90</td>
<td>39.8</td>
<td>-7.0</td>
<td>RA55</td>
</tr>
<tr>
<td>B30</td>
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<td>34.2</td>
<td>-5.3</td>
<td>RB30</td>
</tr>
<tr>
<td>B40</td>
<td>8.33</td>
<td>35.9</td>
<td>-8.0</td>
<td>RB40</td>
</tr>
<tr>
<td>B55</td>
<td>9.28</td>
<td>42.6</td>
<td>-12.3</td>
<td>RB55</td>
</tr>
<tr>
<td>C30</td>
<td>7.70</td>
<td>37.2</td>
<td>-0.1</td>
<td>RC30</td>
</tr>
<tr>
<td>C40</td>
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<td>48.3</td>
<td>-7.1</td>
<td>RC40</td>
</tr>
<tr>
<td>C55</td>
<td>11.79</td>
<td>58.8</td>
<td>-12.8</td>
<td>RC55</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ref.</th>
<th>( \frac{P_f}{s_u^{PS}} )</th>
<th>( \beta \degree )</th>
<th>( \delta_f \degree )</th>
<th>Analytical</th>
</tr>
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<tr>
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</tr>
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<td>36.4</td>
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<td>RA40R</td>
</tr>
<tr>
<td>A55R</td>
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<td>41.6</td>
<td>-8.6</td>
<td>RA55R</td>
</tr>
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<td>B30R</td>
<td>9.72</td>
<td>34.3</td>
<td>-3.2</td>
<td>RB30R</td>
</tr>
<tr>
<td>B40R</td>
<td>10.86</td>
<td>40.9</td>
<td>-10.3</td>
<td>RB40R</td>
</tr>
<tr>
<td>B55R</td>
<td>12.11</td>
<td>43.6</td>
<td>-15.8</td>
<td>RB55R</td>
</tr>
<tr>
<td>C30R</td>
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<td>34.1</td>
<td>+8.1</td>
<td>RC30R</td>
</tr>
<tr>
<td>C40R</td>
<td>12.10</td>
<td>45.3</td>
<td>-4.3</td>
<td>RC40R</td>
</tr>
<tr>
<td>C55R</td>
<td>14.38</td>
<td>56.7</td>
<td>-13.4</td>
<td>RC55R</td>
</tr>
</tbody>
</table>

| Ref. | \( \frac{P_f}{s_u^{PS}} \) | \( \frac{F_r}{B s_u^{PS}} \) |
|------|-----------------|----------------|-------------|
| A30R | 7.30            | 2.13           |
| A40R | 7.42            | 1.61           |
| A55R | 7.88            | 1.29           |
| B30R | 9.35            | 2.95           |
| B40R | 10.49           | 3.28           |
| B55R | 11.02           | 2.86           |
| C30R | 10.28           | 0.69           |
| C40R | 15.53           | 3.48           |
| C55R | 20.80           | 3.70           |

Table 7.2: Final Footing Pressures \( \left( \frac{P_f}{s_u^{PS}} \right) \), Load Spread Angles \( \beta \) , Footing Base Friction Angles \( \delta_f \) and Analytical Reinforcement Forces \( \left( \frac{F_r}{B s_u^{PS}} \right) \)
From the finite element results presented in Figure 7.1 and Table 7.2 it is clear that increasing the fill strength increases the load bearing capacity of the two-layer soil structure for both the unreinforced and reinforced cases. The amount of improvement in $P_f$ obtained by increasing $\phi_{ps}$ from $30^\circ$ to $55^\circ$ is greater for the thicker fill layer ($\approx 50\%$ improvement for $D_f = 0.5 \, m$), compared to the thinner layer ($\approx 9\%$ improvement for $D_f = 0.15 \, m$). These percentages of improvement, due purely to the increased fill strength, appear to be independent of whether the system is unreinforced or reinforced. However, the actual presence of reinforcement has more effect for the thin fill layers than for the thick layers, an average of $35\%$ strengthening is achieved by including reinforcement for $D_f = 0.15 \, m$, compared to the $25\%$ average improvement for $D_f = 0.5 \, m$. Clearly, for large fill thicknesses a greater amount of benefit maybe obtained by increasing the fill strength, than is acquired by including reinforcement, but for thin fill layers the reverse applies.

Increasing the fill depth from $150 \, mm$ to $500 \, mm$ has a small effect on the bearing capacity of the two-layer soil system for unreinforced weak fills ($\approx 6\%$ improvement for $\phi_{ps} = 30^\circ$), but an increased influence for increasing fill strengths ($\approx 49\%$ improvement for $\phi_{ps} = 55^\circ$). Interestingly, in the reinforced analyses increasing the thickness of the weak fill actually reduces the bearing capacity slightly ($\approx -2\%$), although positive improvements are made for the stronger fills ($\approx 34\%$ improvement for $\phi_{ps} = 55^\circ$). Evidently increasing the fill depth significantly increases the bearing capacity for fills of medium strength and above, in this case for $\phi_{ps} \geq 40^\circ$.

For an increase in $D_f$ the calculated angles of friction at the base of the footing, $\delta_f$, generally tend to increase (and similarly the outward acting footing frictional forces) for the stronger fill strengths, but for medium $\phi_{ps}$ they remain comparatively constant and for the weaker fills $\delta_f$ decreases. Furthermore, this trend is somewhat exaggerated by the presence of reinforcement to the extent that the analysis of the reinforced thick weak fill (run C30R) develops an inward acting footing frictional force, $\delta_f = +8.1^\circ$, which significantly reduces the analytical reinforcement force.

Comparing the finite element results of $P_f$, given in Table 7.2, to the equivalent analytical solutions, it is apparent that the finite element predictions are larger for the thin
fill, but for medium $D_f$ they are about equal (or a little higher) and for thick $D_f$ the finite element results are smaller than the analytical. This discrepancy between the finite element and analytical results is particularly noticeable for the analytical runs RC55 and RC55R. The differences in $P_f$ between the finite element and analytical results is a direct consequence of the erroneous assumption made in the Houlsby et al. (1989) method of a constant uniform distribution of normal stress within the load spread area at the fill base. The actual upper interface normal stress distributions (shown in Figure 7.4) vary substantially, and are more spread out for $D_f = 0.5 \ m = 2.0B$ than for $D_f = 0.15 \ m = 0.6B$, which consequently produces larger values of load spread angle (by the 95% limit method) and therefore gives spuriously high values of analytical $P_f$ for the thicker fills. It is noteworthy that for the thick weak fill analysis (RC30R) the analytical method actually predicts a bearing capacity type failure of the fill, because of the large $\beta$ value.

The variations in the load spread angle ($\beta$) for increases in the fill strength and fill depth are presented in Table 7.2 and shown in Figures 7.2 a) and b) respectively. The load spread angle is extremely variable with $\phi_{ps}$ and $D_f$, but interestingly it is relatively insensitive to the presence of reinforcement. In Figures 7.2 a) and b) $\beta$ increases only slightly with an increasing $\phi_{ps}$ for $D_f \leq 1.2B$ (i.e. the increase between $\beta_{A30}$ and $\beta_{A55}$ is only 7%), but for $D_f > 1.2B$ then $\beta$ is a strong function of the friction angle (i.e. the increase between $\beta_{C30}$ and $\beta_{C55}$ is 58%).

These variations in $\beta$ for changes in $D_f$ and $\phi_{ps}$, observed in the finite element analyses, can be compared to those variations seen in the experimental tests of Love (1984) and to those variations predicted by the bearing capacity theory proposed by Meyerhof (1974).

Love (1984) recorded measurements of the area along the clay surface effected by the footing load (which is an indication of the load spread angle), for different fill thicknesses, in his experimental two-layer soil tests. The total effective footing width on the clay ($2B'$) was defined as the distance along the clay surface between stationary points in the displacement vector plots. These measurements and the related load spread angle, derived by equation (7.2), are presented in Table 7.3 for each $D_f$ (where $B$ is the footing half width = 37.5 mm).
Burd (1986) back-analysed the experimental tests by Love (1984) and established a best fit friction angle of $\phi_{PS} = 36^\circ$. With reference to Figure 7.2 a), a friction angle of 36$^\circ$ would be expected to give an increasing $\beta$ for increasing values of $D_f > 1.2 B$. However, the experimentally determined $\beta$ values in Table 7.3 seem to be insensitive to the fill depth and, moreover, consistently higher for the reinforced case compared to the unreinforced case, unlike the numerically established load spread angles. This difference of the variations in $\beta$ for changes in $D_f$, with and without reinforcement, is because Love (1984) interpreted the length $B'$ from the measured displacements within the clay, whereas the finite element $\beta$ in Table 7.2 is calculated from the computed normal stress distributions within the fill.
<table>
<thead>
<tr>
<th>Nominal Clay Strength (kPa)</th>
<th>Reinforced</th>
<th>Unreinforced</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>124 (26.1°)</td>
<td>110 (19.3°)</td>
</tr>
<tr>
<td>9</td>
<td>108 (18.3°)</td>
<td>106 (16.7°)</td>
</tr>
<tr>
<td>14</td>
<td>105 (16.7°)</td>
<td>97 (12.4°)</td>
</tr>
<tr>
<td>6</td>
<td>134 (21.5°)</td>
<td>132 (20.8°)</td>
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<tr>
<td>9</td>
<td>137 (22.5°)</td>
<td>129 (19.8°)</td>
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<td>14</td>
<td>110 (13.1°)</td>
<td>129 (19.8°)</td>
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<td>8</td>
<td>163 (23.7°)</td>
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<td>9</td>
<td>142 (18.5°)</td>
<td>139 (17.7°)</td>
</tr>
<tr>
<td>14</td>
<td>139 (17.7°)</td>
<td>130 (15.4°)</td>
</tr>
</tbody>
</table>

Table 7.3: Values of $2B'$ (mm) and $\beta$, estimated from equation (7.2), at a Footing Penetration of 25 mm = $2B'/3$

(After Love (1984))

\[
\beta = \tan^{-1} \left( \frac{B' - B}{D_f} \right) \tag{7.2}
\]

Clearly, the load spread angle, which is only a convenient simplification of the real loading situation anyway, is highly sensitive to the particular method of interpretation. However, it should be noted that the experimental $\beta$, in Table 7.3, does tend to reduce with increased clay strength, which is an important relationship discussed further in Chapter 10.

Meyerhof (1974) predicted the ultimate bearing load ($F_u$), of a strip footing resting on a dense sand layer overlying soft clay, to be:

\[
F_u = 2B (\pi + 2) s_{u}^{PS} + 2P_{p} \sin \delta \tag{7.3}
\]

where $B$ is the footing half width and $P_p$ is the total passive earth pressure, inclined at an average angle $\delta$ acting upwards on a vertical plane through the footing edge, Figure 7.3.

The failure surface in the fill was originally thought to be of a logarithmic spiral shape (Meyerhof (1974)), but this was later modified to a more uniform truncated pyramidal shape by Hanna and Meyerhof (1980), which is assumed here to have a constant inclination of $\beta$ from the vertical, as shown in Figure 7.3.
Figure 7.3: Failure of Soil below Footing on Dense Sand above Soft Clay
(After Meyerhof (1974))

The total passive earth pressure is given by Meyerhof (1974) as:

$$P_p = \frac{\gamma_f D_f^2 K_s \tan \phi_{ps}}{2 \sin \delta} \quad (7.4)$$

where $K_s$ is the coefficient of punching shearing resistance on the vertical plane through the footing edge, determined from the corresponding earth pressure coefficients, and $\gamma_f$ is the unit weight of the fill.

By force equilibrium it is shown that:

$$P_p \sin \delta = l_x (\pi + 2) s_u^{PS} \quad (7.5)$$

where $l_x$ is the horizontal distance, along the clay surface, between the boundary of the load spread area and the vertical plane through the footing edge (see Figure 7.3).

Thus, by substituting equation (7.4) into equation (7.5), the assumed load spread angle, $\beta$, is given by:

$$\tan \beta = \frac{l_x}{D_f} = \frac{P_p \sin \delta}{(\pi + 2) s_u^{PS} D_f} = \frac{\gamma_f D_f K_s \tan \phi_{ps}}{2 (\pi + 2) s_u^{PS}} \quad (7.6)$$
This new relationship in equation (7.6), derived from the Meyerhof (1974) theory, does not predict the correct \( \beta \) angles as given in Table 7.2, but it does show how \( \beta \) varies with the fill depth, fill friction angle and clay strength. The coefficient \( K_i \) is shown by Meyerhof (1974) to increase rapidly with \( \phi_{ps} \). Thus, the variations in \( \beta \) can be tabulated as:-

<table>
<thead>
<tr>
<th>By equation (7.6)</th>
<th>Finite Element</th>
<th>Love (1984)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta ) increases with increasing ( D_f )</td>
<td>YES (for ( \phi_{ps} \geq 40^\circ ))</td>
<td>NO</td>
</tr>
<tr>
<td>( \beta ) increases rapidly with increasing ( \phi_{ps} )</td>
<td>YES (for ( D_f &gt; 1.2 , B ))</td>
<td></td>
</tr>
<tr>
<td>( \beta ) reduces with increasing ( s^u_{ps} )</td>
<td>YES (Chapter 10)</td>
<td>YES</td>
</tr>
</tbody>
</table>

Table 7.4: Variations of \( \beta \) by Theoretical, Numerical and Experimental Analyses

7.3.2 Normal Stress Distributions

The distributions of normal stress (\( \sigma \)) along the unreinforced and reinforced interfaces of the eighteen different two-layer soil systems listed in Table 7.1, are shown in Figure 7.4, where \( \sigma \) is non-dimensionalised by the plane strain, undrained, clay shear strength (\( s^u_{ps} \)) and the distance from the centre-line (\( x \)) is non-dimensionalised by the footing half width (\( B = 0.25 \, m \)). The range of calculated load spread angles (\( \beta \)) for changes in fill strength, at a particular fill depth, are also given within Figure 7.4.

For each of the unreinforced runs the normal stresses at the centre-line are calculated as \( 5.5 \, s^u_{ps} \) for \( D_f = 0.6 \, B \) and \( 5.7 \, s^u_{ps} \) for \( D_f \geq 1.2 \, B \) approximately, which is between 5\% and 7\% more than the ultimate bearing capacities, given by \( (\pi + 2) \, s^u_{ps} + \gamma \, D_f \), for each fill depth. This small amount of additional peak loading, above the ultimate bearing capacity, is believed to be a consequence of the peripheral load distribution outside of the established load spread area, as described in Section 6.3.5 and Figure 6.13.

For unreinforced thin fills \( \sigma \) maintains a relatively constant distribution within the loaded area independent of the friction angle, but as \( D_f \) increases the distribution becomes
Figure 7.4: Normal Stress ($\sigma / s_{ps}^*$) along Interface for Varying Fill Depths ($D_f$) and Friction Angles, (at $\delta = 0.6 \, B$)
increasingly spread out by an amount that depends on the friction angle, (seen particularly clearly for \( D_f = 0.5 \ m \)). This is because of the presence of large outward acting shear stresses (illustrated in Figure 7.5) which reduce the normal stresses, particularly for the thicker fill.

The total normal force acting at the unreinforced interface and the area of the load spread, both increase for increasing \( \phi_{ps} \) and as the fill depth increases this effect becomes even more appreciable, i.e. the normal force for run A55 is 6% greater than for run A30, while for run C55 the normal force is 33% greater than for run C30. This is the same effect as seen in the pressure-displacement responses, Figure 7.1.

In the reinforced analyses the lower normal stress distributions each begin on the centre-line at approximately \( 6 u_{ps}^p \), which is due to the increased load capacity of the clay created by the inward acting shear stresses (Figure 7.5) outside of the vertically loaded area (Figure 6.16 Mode 2). The lower normal stresses maintain a more uniform distribution within the load spread area, in comparison to the equivalent unreinforced analyses, as predicted by plasticity theory. The extent of the load spread area increases for increasing \( \phi_{ps} \) and for increasing \( D_f \), as in the unreinforced cases. The normal stress distributions on the upper reinforcement surface are different from those on the lower surface, especially for the thin fill thickness, but this effect reduces for increasing \( D_f \) where the upper normal stresses reach progressively smaller maximum peak values, i.e. the effect of reinforcement on the normal stresses is much more distinctive for the thin fills than for the thick fills. Clearly, the large displacement reinforcement mechanisms, such as the tensioned membrane mechanism, contribute less to the overall reinforcing effect as the fill depth increases, whereas the small displacement mechanisms, such as changes in load spread angle, contribute more.

**7.3.3 Shear Stress Distributions**

As with the normal stresses, the distributions of shear stress, \( \tau \), along the unreinforced and reinforced interfaces are shown in Figure 7.5.
Figure 7.5: Shear Stress ($\tau / s_{u}^{PS}$) along Interface for Varying Fill Depths ($D_f$) and Friction Angles, (at $\delta = 0.6 \, B$ )
For the unreinforced analyses inward acting shear stresses of about \(-0.8 \, s_u^{PS}\), maximum, develop directly under the footing for \(D_f = 0.15 \, m\), but as the fill depth increases these reduce significantly. These inward shear stresses, within the vertically loaded area, do not contribute globally to the bearing capacity of the clay, but are significant in that they contradict the assumption in the analytical method, which predicts large outward acting shear stresses. At approximately \(x = 1.2 \, B\) the shear stresses reverse direction and as \(D_f\) increases these outward acting stresses become greater (peak \(\tau = 1 \, s_u^{PS}\)) and distribute across a larger area before finally diminishing to zero. It is these large outward shear stresses, away from the loaded area beneath the footing, that are responsible for the reduction in the normal stresses, i.e. the shear stress effect.

In the reinforced analyses the shear stresses on the lower reinforcement surface are virtually insignificant directly under the footing regardless of the fill depth. However, further along the clay surface \(\tau\) becomes quite a large inward acting stress for the thin fill layer \((D_f = 0.15 \, m)\) reaching a peak of \(-1.2 \, s_u^{PS}\) beyond the load spread (greater than \(s_u^{PS}\) because of the discretization error, as explained in Section 5.4.5), which improves the load bearing capacity of the clay by approximately 11% (Figure 6.16). As \(D_f\) increases these inward clay shear stresses diminish and remain relatively small throughout the interface length, illustrating the reduced shear stress mechanism of the reinforcement for thick fills.

The shear stress distributions on the upper reinforcement surface for \(D_f = 0.15 \, m\) vary between large inward acting \(< -2 \, s_u^{PS}\) under the footing, to large outward acting \(> 2 \, s_u^{PS}\). As \(D_f\) increases the inward shear stresses reduce considerably, leaving only the outward shear stresses acting on the upper reinforcement surface.

A feature of all the unreinforced and lower reinforced shear stress distributions shown in Figure 7.5 is that they are relatively unaffected by variations in the fill strength, \(\phi_{PS}\). However, the upper reinforced shear stress distributions tend to both increase in outward acting peak value and to shift further along the reinforcement, away from the footing centre-line, for an increase in \(\phi_{PS}\), which is an effect that becomes more pronounced with increasing fill thickness.
7.3.4 Reinforcement Tension Distributions

The distribution of tension in the reinforcement, $F_r$, along its length is shown in Figure 7.6 for each fill depth.

With increasing variations in the fill strength, the peak reinforcement tensions increase marginally and shift slightly away from the footing centre-line. This effect becomes more exaggerated with decreasing $D_f$, but the more prominent feature seen in Figure 7.6 is that the centre-line tension increases appreciably as both the fill layer thickens and $\phi_{PS}$ decreases. This is because the magnitude of the outward shear force applied by the fill increases as $D_f$ increases and it moves nearer to the footing centre-line as $\phi_{PS}$ reduces (Figure 7.5). Consequently the reinforcement sustains a greater amount of force, in the area directly under the footing, as $D_f$ increases.

The maximum reinforcement tensions predicted using the Houlsby et al. (1989) analysis, Table 7.2, are lower than the peak tensions shown in Figure 7.6, and are, in fact, better approximations of the centre-line tensions. It is suggested that the reason for this is two-fold. Firstly, outward shear stresses act on the underside of the reinforcement for distances of up to $5B$ from the footing centre-line (Figure 7.5). This results in a build up of reinforcement force with distance towards the footing centre-line. For the case of $D_f = 0.6B$ this results in a tension of about 1.9 (non-dimensional units) at $x = 2B$. At this point the outward acting shear stresses, on the upper reinforcement surface, combine to produce a peak tension of about 4.0 units. It is the tension associated purely with these outward acting shear stresses (in this case about 2.1 units) which should be compared with the limit-equilibrium results (in this case 2.13 units for $\phi_{PS} = 30^\circ$). This particular example is a very good agreement, although the comparison gets slightly worse with increasing friction angle. For the thicker fill, $D_f = 2.0B$, the lower shear stresses are very small and do not significantly contribute to the reinforcement tension, consequently the analytical predictions are within just 3% of the peak finite element values of $F_r$, except for run RC30R, which is because of the large inward footing friction force. Secondly, the Houlsby et al. (1989)
Figure 7.6: Reinforcement Tension ($F_r / (B s_u^{ps})$) for Different Fill Depths ($D_f$) and Friction Angles, (at $\delta = 0.6 B$)
method for calculating tensions due to fill shear stresses is non-conservative, it assumes that the fill is in a state of active failure beneath the footing when generally it is not, as discussed in Section 6.5.6.

7.3.5 Slip and Deformation along Interfaces

The relative slip, \( u_r \), along the unreinforced and the upper and lower reinforced interfaces is shown in Figure 7.7, for each fill depth and strength. As \( D_f \) increases, the magnitude of the slip decreases for all fill strengths. The reinforced analyses, however, maintain higher values of \( u_r \), compared to the unreinforced due to the higher footing loads. In the unreinforced cases there is a small amount of negative slip directly below the footing, where the clay is moving out (away from the footing centre-line) from under the fill layer, which creates inward acting shear stresses on the clay surface as shown in Figure 7.5. However, beyond about \( x = 1.5 \, B \) the fill begins sliding out over the clay (positive \( u_r \)) creating outward shear stresses which reduce the clay bearing capacity factor, \( N_c \). This mode of positive slip is sustained over a large distance for the thick fill layer, whereas for the thinner layers the relative slip reverses direction again, but does not generate any significant shear stresses because the normal stresses are small, Figure 7.4.

In the reinforced cases the upper interfaces are subjected initially to negative \( u_r \) and therefore inward shear on the reinforcement, due to the clay dragging the reinforcement outwards, for all fill thicknesses and strengths. However, between approximately \( x = 1.5 \, B \) and \( 3 \, B \) the fill slides out over the reinforcement, generating outward upper shear stresses, Figure 7.5. Concurrently the lower reinforcement interfaces experience negative slip (i.e. inward shear on the clay) as the clay flows away from the centre-line, an effect which diminishes to zero for the thick fill.

The final deformed profiles of the unreinforced and reinforced interfaces are plotted in Figure 7.8. The presence of reinforcement changes the profiles appreciably only in the area of reversed curvature, by reducing the amount of upward heave. As \( D_f \) increases, the amount of vertical displacement at the centre-line reduces slightly (\( \approx 3\% \) for \( D_f = 2 \, B \)), for unreinforced and reinforced. The noticeably lower centre-line displacements for runs C30 and C30R are the consequence of a bearing capacity type failure operating in the fill.
Figure 7.7: Relative Slip ($U_r$) along the Interfaces
Figure 7.8: Final Deformed Shape of the Interfaces
7.4 General Conclusions

The effect of reinforcement seems to be greater for thin fills than for thick fills. The pressure-displacement responses, Figure 7.1, show that for a fill depth of 0.6B the effect of reinforcement is very marked, whereas increasing the fill friction angle has little effect. For thick fills (Df = 2B) changes in the friction angle have a much greater effect than including reinforcement. This also shows up well in the distributions of the normal stress, Figure 7.4. At large values of Df it may be more effective to increase f rather than to use reinforcement (although the presence of reinforcement may, indirectly, improve compaction and hence f).

The normal stress plots illustrate well the effect of increasing the fill thickness on the width of the loaded area on the clay surface. For thin fills the stresses drop sharply to zero at a point that is roughly independent of friction angle. For thick fills, however, the normal stress distributions are more smooth and extend laterally by an amount that depends on the fill friction angle. This aspect of the results leads to considerable difficulties in interpreting the results using simple load spread type models. Firstly, the diffuse nature of the normal stresses for thick fills results in a value of β which, when used in a conventional load spread analysis, gives loads that are too large (Table 7.2). Secondly, the value of any load spread parameter clearly varies with fill thickness and friction angle in a way that is difficult to quantify. A compromise might be to suggest that a simple load spread analysis might be suitable for, say, Df < B, for thicker fills a different analysis would need to be adopted.

Investigating the normal stresses at the interface of the reinforced two-layer soil system illustrates the reduced role of the large displacement reinforcement mechanisms, such as the tensioned membrane, and the increased role of the small displacement mechanisms, such as a larger load spread area, which occur as the fill depth increases.
CHAPTER 8:

INFLUENCE OF REINFORCEMENT STIFFNESS

8.1 Introduction

The plethora of reinforcement materials that are currently available to design engineers is continually increasing. There is a large variety of different materials, such as polymer grids and synthetic fabrics, each designed to offer different physical capabilities, such as strength, interlock, separation, porosity, transmissivity, etc., depending upon the particular requirement. Naturally this has resulted in the materials possessing a wide range of different stiffnesses, for example Tensar materials have a stiffness of between 350 kN/m to 730 kN/m, whereas Bidim materials have typically between 800 kN/m and 2000 kN/m stiffness. Clearly the stiffness of a geosynthetic material is an important property to consider when designing reinforced two-layer soil structures subjected to footing loads.

The design procedures currently in use for two-layer soil systems are generally based on the use of simple analytical models of behaviour to calculate the response of the structure to a single load application, with the extension for the case of repeated loading made on the basis of empirical correlations, as discussed in the literature review. These design procedures, in general, develop a series of mathematical equations which can be used to assure stability of the two-layer soil system. A large number of variables are introduced into the design equations in an attempt to model accurately the various materials involved. The stiffness of the geosynthetic reinforcement is one such commonly used design variable, see Nieuwenhuis (1977), Giroud and Noiray (1981), Sowers et al. (1982), Giroud et al. (1984), De Groot et al. (1986), Sellmeijer et al. (1982), and Sellmeijer (1990). However,
it is not a parameter considered in all of the currently used design methods, for instance Houlsby et al. (1989) and Houlsby and Jewell (1990), where the inherent assumption is made that the geotextile stiffness is sufficient to provide 'full reinforcement'.

The purpose of this study is to decide whether the concept of 'full reinforcement' is useful and if so, how stiff the reinforcement needs to be in order to achieve it.

**8.2 Parametric Study**

This study is based on four large displacement plane strain finite element calculations using the 'central parametric analysis' and the material properties given in Appendix 6A, with a variety of reinforcement stiffnesses.

The reinforced 'central parametric analysis' (run B40R) possesses a reinforcement stiffness of 300 kN/m, which is typical of a proprietary polymeric geogrid material. Geosynthetic membranes of stiffnesses 30 kN/m (run B40A) and 3 000 kN/m (run B40C) would be thought of as being 'flexible' and 'stiff' respectively. The equivalent unreinforced two-layer soil system (run B40) is also included in the study.

The small displacement Houlsby et al. (1989) analytical predictions equivalent to the unreinforced and reinforced finite element 'central analyses' are presented for comparison.

**8.2.1 Load-Displacement Response**

The footing pressure-displacement responses for each of the four large strain finite element runs and the two 'central' comparative analytical predictions, are illustrated in Figure 8.1, where \( P \) is the average value of vertical stress beneath the footing and \( \delta \) is the vertical footing displacement. The full set of results are presented in Table 8.1, where 'U' signifies the unreinforced and 'R' the reinforced analytical calculations.
Figure 8.1: Pressure-Displacement Response for Different Reinforcement Stiffnesses ($J$)

<table>
<thead>
<tr>
<th>Ref.</th>
<th>$J$</th>
<th>$P_f/s_u^{PS}$</th>
<th>$\delta_f$</th>
<th>$\beta$</th>
<th>Ref.</th>
<th>$J$</th>
<th>$P_f/s_u^{PS}$</th>
<th>$F_\gamma/(B s_u^{PS})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>B40</td>
<td>0</td>
<td>8.33</td>
<td>$-8.0^\circ$</td>
<td>35.9$^\circ$</td>
<td>RB40</td>
<td>U</td>
<td>6.56</td>
<td>-</td>
</tr>
<tr>
<td>B40A</td>
<td>30</td>
<td>9.17</td>
<td>$-9.8^\circ$</td>
<td>37.1$^\circ$</td>
<td>RB40A</td>
<td>R</td>
<td>9.81</td>
<td>3.02</td>
</tr>
<tr>
<td>B40R</td>
<td>300</td>
<td>10.86</td>
<td>$-10.3^\circ$</td>
<td>40.9$^\circ$</td>
<td>RB40R</td>
<td>R</td>
<td>10.49</td>
<td>3.28</td>
</tr>
<tr>
<td>B40C</td>
<td>3000</td>
<td>13.24</td>
<td>$-7.8^\circ$</td>
<td>36.7$^\circ$</td>
<td>RB40C</td>
<td>R</td>
<td>9.74</td>
<td>2.65</td>
</tr>
</tbody>
</table>

Table 8.1: Run References for the Finite Element and Analytical Results Shown in Figure 8.1

Clearly the reinforcement has little influence on the elastic behaviour of the system, but an increasing influence as the footing displacement increases and plastic soil deformations start to occur. At footing displacements larger than $\delta = 0.1B$ a greater improvement in the load bearing capacity of the system is obtained in-line with the increased stiffness of the reinforcement. The gradients of the pressure-displacement curves for the reinforced runs, increase with reinforcement stiffness due to the ‘tensioned membrane’ effect, where the tensions in the reinforcement combine with the membrane curvature to
reduce the normal stresses applied to the subgrade (illustrated in Figure 8.2), thus increasing the footing load for a given footing displacement. This effect increases with increasing footing displacement and reinforcement stiffness.

The amount of improvement in the final footing pressure $P_f$ at $\delta = 0.6\, B$, acquired through increasing the stiffness of the reinforcement, is not proportional to the actual stiffness value. Incorporating a flexible reinforcement with a stiffness of 30 kN/m (B40A) increases the load by 10% above the unreinforced capacity. Increasing the stiffness tenfold increases this improvement to 30%, and a further tenfold stiffness increase takes this figure to 60%. Clearly the increase in load capacity is not proportional to the stiffness and is a case of diminishing returns, suggesting that the use of very stiff reinforcement material in two-layer soil systems is unlikely to be economical. Furthermore, this contradicts the theory of any analytical methods which predict a linear relationship between the value of reinforcement stiffness and the bearing capacity improvement, e.g. Giroud and Noiray (1981) and Sowers et al. (1982).

The analytical method underestimates $P_f$ for the unreinforced case (RB40), because it predicts a large contribution by the footing frictional force to the assumed outward acting shear stresses at the base of the fill layer, when in fact the shear stresses are inward acting in the area beneath the footing (Figure 8.2). For the cases with a flexible (RB40A) and a moderately stiff (RB40R) reinforcement the analytical predictions of the final footing pressures are reasonably comparable to the finite element values, with less than a 6.5% difference. However, the analysis for the stiff reinforcement (RB40C) substantially under-predicts $P_f$, because the Houlsby et al. (1989) method does not consider the influence of the large displacement tensioned membrane mechanism, which becomes significant for this high stiffness case, Figure 8.2.
8.2.2 Normal and Shear Stress Distributions

The normal (σ) and shear (τ) stresses, extracted from the appropriate interface element Gauss points, that are acting along the upper and lower reinforcement surfaces at the final footing displacement of \( \delta = 0.6 \, B \) are plotted in Figure 8.2.

![NORMAL STRESS](image1)

![SHEAR STRESS](image2)

**Figure 8.2:** Normal Stress \( (\sigma / S_u^{PS}) \) and Shear Stress \( (\tau / S_u^{PS}) \) along Interface for Varying Reinforcement Stiffnesses \( (J) \), \( (\text{at } \delta = 0.6 \, B) \)

Comparison of the upper and lower surface normal stress distributions indicate that within the area of approximately \( x = 0 \) to \( 1.5 \, B \) the reinforcement is subjected to different magnitudes of normal stress. This difference in \( \sigma \) is due to the tensioned membrane effect which increases with the reinforcement stiffness. The magnitude of the normal stresses

8-5
acting on the subgrade immediately beneath the footing is approximately constant and it is suggested that in this region the clay is being subjected to a bearing capacity type failure. If no shear stresses were acting on the clay surface then the magnitude of the normal stresses in this region would be expected to be a constant value of \((\pi + 2) s_u^{PS} + \gamma_f D_f\) as given by equation (6.3). However, the presence of the outward acting shear stresses has the effect of reducing the normal stresses, particularly for the runs B40 and B40A.

The shear stress distributions along the upper reinforcement surface show that for each value of reinforcement stiffness the outward shear stresses induced by the fill on the reinforcement reach a peak at a horizontal distance \(x\) of about 1.5 \(B\) from the footing centre-line and that as the reinforcement stiffness increases then the magnitude of these shear stresses also increases. It is suggested that this effect is caused partly by the greater total footing load, which is needed to achieve \(\delta = 0.6B\), and partly by the reinforcement constraining the base of the fill from displacing laterally by an amount that increases as the reinforcement stiffness increases. This fill restraint mechanism, discussed in Chapter 2, thus causes an increase in the lateral stresses induced at the top of the fill, as seen in the plots of the principal stress directions, Figures 8.6.

Similar increases in shear stress at the upper reinforcement interface, for increases in reinforcement stiffness, were found by Hird and Kwok (1989) and (1990). They propose that high lateral stresses develop near the top of the fill embankment due to its "bending" or "arching", comparable to Figure 8.6 b). Which, for a very high reinforcement stiffness (i.e. 15 000 kN/m\(^2\)), become excessively large, leading to an increase in the upper interface shear stresses and hence the total reinforcement force.

The shear stress distributions along the lower reinforcement surface, immediately beneath the footing, show that the clay subgrade is subjected to a small amount of inward acting stress for all values of reinforcement stiffness. However, for values of \(x\) greater than 1.5 \(B\) the shear stresses for the unreinforced case (B40) and the flexible reinforcement case (B40A), reverse direction and become significant outward acting stresses (\(\tau_{peak} = 1 s_u^{PS}\)) which reduce the normal stresses acting on the clay at failure (refer to Figure 6.16 Mode 4). These detrimental outward shear stresses arise from the transmission of shear stresses at the
base of the fill, highlighting the inadequacy of very flexible membrane materials. For the stiffer reinforcements the clay layer is subjected to inward acting shear stresses almost throughout, i.e. the beneficial ‘shear stress mechanism’ of the reinforcement. The magnitudes of these inward shear stresses increase with increasing reinforcement stiffness, which is caused by the reinforcement restraining the clay as it endeavours to flow plastically away from the footing centre-line, see Figure 8.4.

There appears to be no significant change in the load spread angle ($\beta$) for increasing reinforcement stiffness, judging from the data in Table 8.1 and the plot of normal stresses at the fill base. However, there is an observed small improvement in the load distribution on the clay surface, but this does not contribute to the overall reinforcing effect to the same extent as the large displacement tensioned membrane mechanism.

The theory presented in the Houlsby et al. (1989) limit state design method assumes that none of the outward acting shear stresses at the base of the fill layer are transmitted onto the clay surface and that the entire reinforcement force is due solely to the fill shear stresses. It is also assumed that to achieve effective reinforcement the geotextile must be capable of providing a sufficient force at both ultimate capacity and small deformations, which will require the reinforcement to have a certain stiffness. However, it has been shown in the plots of shear stress distribution, Figure 8.2, that the shear stresses vary between inward and outward acting along the length of both the upper and lower membrane surfaces. Therefore, it can be assumed that the reinforcement tension is actually due to the sum of the shear forces that act on both the upper and lower surfaces of the reinforcement, which are found by integrating the shear stresses. This method of estimating the reinforcement force was used in the small strain finite element study of the influence of the reinforcement stiffness in two-layer soil systems, undertaken by Burd and Brocklehurst (1990). It was found that for low reinforcement stiffnesses, $< 60 \text{kN}/\text{m}$, a proportion of the shear force induced by the fill was sustained directly by the clay, but as the reinforcement stiffness increased then the shear force in the clay reduced. However, for very large reinforcement stiffnesses, $> 3000 \text{kN}/\text{m}$, the subgrade shear force began to increase, but acting inwards rather than outwards as it had for the smaller stiffnesses. This had the net result that the shear stresses at the reinforcement-clay interface continued to increase the force in the reinforcement.
8.2.3 Reinforcement Tension Distribution

The tensions generated in the reinforcement at the end of each calculation are plotted in Figure 8.3 in conjunction with the comparative analytical 'central' prediction, RB40R.

![Graph showing reinforcement tension distribution](image)

**Figure 8.3: Reinforcement Tension ($F_r / (B \cdot s_{up})$) for Different Reinforcement Stiffnesses ($J$), (at $\delta = 0.6B$)**

An important feature of these results is that increasing the reinforcement stiffness causes a corresponding increase in the magnitudes of the forces induced in the reinforcement with only a small improvement in the footing pressure-displacement response, Figure 8.1. A similar result was also found in the Burd and Brocklehurst (1990) small strain reinforced two-layer soil system calculations for varying reinforcement stiffness.

Although the analytical values of the maximum reinforcement tension for the flexible (RB40A) and stiff reinforcements (RB40C), Table 8.1, are very different from the finite element results, because there is no consideration of the shear force applied to the reinforcement by the clay, the $F_r$ value for the 'central' analysis is notably only slightly smaller than the peak value for run B40R, Figure 8.3.

8.2.4 Slip and Deformation along Soil-Reinforcement Interface

The relative slip, $u_r$, along the upper and lower reinforcement surfaces is illustrated in Figure 8.4 for each analysis. Positive slip indicates the upper interface surface moving to the right relative to the lower surface. As the reinforcement stiffness increases the magnitude of the upper relative slip increases also. Note the large amount of slip on the underside of the
reinforcement for run B40C, due to the clay flowing away from the centre-line, which generates the high inward shear stresses in Figure 8.2. Furthermore, the lower slip for run B40A is uncharacteristically positive at about $x = 2.3B$, because the reinforcement is not stiff enough to restrain the fill, causing outward shear stresses on the clay surface. The slip along the unreinforced interface is shown in Figure 6.23 a).

![Figure 8.4: Relative Slip ($u_r$) along Interfaces](image)

Figure 8.4: Relative Slip ($u_r$) along Interfaces

The final deformed profiles of the unreinforced and reinforced soil interfaces are shown in Figure 8.5. Clearly the reduction in the magnitude of the displacements at the interface for the reinforced cases, compared to the unreinforced case, is dependent upon the stiffness of the reinforcement. As $J$ increases the membrane profile changes significantly, which contradicts the theory of any analytical methods which assume that the reinforcement maintains a constant deformed shape regardless of the reinforcement stiffness (e.g. the parabola assumed by Giroud and Noiray (1981)).

![Figure 8.5: Final Deformed Shape of the Membrane](image)

Figure 8.5: Final Deformed Shape of the Membrane
8.2.5 Principal Stress Directions

The principal stress directions in the fill layer for the flexible (B40A) and stiff (B40C) reinforcement analyses are plotted in Figure 8.6 and for the unreinforced (B40) and moderately reinforced (B40R) analyses in Figure 6.25 a). By comparing these figures it is evident that not only does the reinforcement cause a change in the principal stress directions, but also the magnitude of the principal stresses increases as the reinforcement stiffness increases, and an arching effect progressively develops beneath the footing.

Hird and Kwok (1989) and (1990) report that placing reinforcement at the base of an embankment can significantly reduce deformations in the foundation provided it is sufficiently stiff and strong, but that the benefit of increasing the reinforcement stiffness follows a "pattern of strongly diminishing returns". They conclude that for very stiff reinforcement the probability of arching in the embankment is increased, which would induce increased reinforcement force without any increase in stability of the fill. They predict that very little benefit is obtained by increasing the reinforcement stiffness beyond about 7 000 kN/m. Clearly, this same effect is also operating here, Figures 8.3 and 8.6.

8.3 General Conclusions

This large displacement parametric study suggests that increasing the reinforcement stiffness of a reinforced two-layer soil system can result in significant improvements in the load-displacement response of the footing, compared to that of an unreinforced system. A 60% greater final load capacity was achieved with a stiff reinforcement, compared to 10% improvement for a flexible one, at a footing displacement of 0.6 B. In contrast, the maximum improvement obtained with $J = 15 000$ kN/m in a small displacement finite element calculation was only 15%, see Burd and Brockenhurst (1990).

As the reinforcement stiffness is increased the large displacement tensioned membrane mechanism provides an increasing contribution to the load improvement, whereas $\beta$ remains relatively constant. However, for static loading there is little benefit to be gained from using excessively stiff reinforcement, since this generates an arching effect in the fill and induces large shear stresses between the soil and reinforcement, which do not in themselves contribute to an improvement in the static bearing capacity of the system and also give rise to large values of reinforcement force. It is possible, nevertheless, that these additional
stresses may improve the cyclic behaviour of the system. It seems probable that in practice the use of very stiff reinforcement is unlikely to be economical, for the returns gained in improved static loading capacity.

Figure 8.6: Principal Stress Directions in the Fill at $\delta = 0.6 B$
CHAPTER 9:

INFLUENCE OF REINFORCEMENT LENGTH

9.1 Introduction

There are many analytical design methods for reinforced two-layer soils systems which advocate the need for membrane anchorage. The design methods either assume that the membrane is fixed at some arbitrary points beyond the load spread area, or stipulate that a minimum membrane length must be maintained outside of the area directly influenced by the load. This is so that simplifying assumptions can be made in order to calculate the tension in the membrane and hence the contribution of the tensioned membrane mechanism. The design methods and studies by Giroud and Noiray (1981), Giroud et al. (1984), Sellmeijer et al. (1982), De Groot et al. (1986), Bourdeau (1989) and Dembicki and Jermolowicz (1991), all specify membrane anchorage for the reasons which are elaborated upon below.

Giroud and Noiray (1981) and Giroud et al. (1984) determine the membrane curvature and hence estimate the reinforcement tension by assuming that the membrane is pinned at the limits of the loaded area. De Groot et al. (1986) describes a design method based on Sellmeijer et al. (1982) which specifies a required "anchorage length of geotextile" under the fill, over which the stress that has developed by the strain in the heavily deformed part of the membrane, will be reduced to zero due to friction on both sides of the geotextile. Bourdeau (1989) uses an iterative finite difference scheme to compute the tensile forces in the membrane by resolving both the vertical fill stress and the horizontal component of the frictional stress at the upper membrane interface. A minimum "active" length of
reinforcement is then evaluated to ensure full frictional anchorage such that the tension is zero at the free end. Dembicki and Jerinolowicz (1991) calculate a minimum "length of anchorage", thought necessary to prevent slip at the reinforcement ends, by a force equilibrium equation which is dependent upon the fill-geotextile angle of friction and the coefficient of clay-geotextile adhesion, for a known maximum reinforcement tension.

However, from the results of the parametric study conducted in this chapter it is clear that the reinforcement does not need to be anchored to obtain an improvement in the structural performance of a two-layer soil system, nor indeed is it essential that the reinforcement covers any large width of the fill base.

### 9.2 Parametric Study

A facility exists within the finite element program OXFEM which allows certain reinforcement membrane elements to be prescribed a stiffness \((J)\) of zero, while other membrane elements possess a real stiffness value, thus effectively changing the reinforcement length. Using this method, six large displacement plane strain finite element runs were completed with various reinforcement lengths \((L)\) taken from the footing centre-line, Figure 9.1, using the finite element mesh B, Figure 5.2, and the material properties as given in Appendix 6A.

![Figure 9.1: Simplified Cross Section of a Reinforced Two-Layer Soil System](image)

9-2
9.2.1 Load-Displacement Response

The calculated footing pressure-displacement responses for each of the six finite element runs and the two 'central' analytical runs are shown in Figure 9.2 and the full set of results are presented in Table 9.1, where \( P \) is the average vertical footing pressure, \( \delta \) is the vertical footing displacement, \( \beta \) is the load spread angle through the fill and \( \delta_f \) is the angle of friction at the footing base.

The results of run B40R, with \( L = 20B \), are listed in Table 9.1 so that they can be compared to those of run B40R5. However, the pressure-displacement response of run B40R is not plotted in Figure 9.2, because it is effectively identical to the response of run B40R5, as shown by comparing Figures 9.2 and 6.10.

![Figure 9.2: Pressure-Displacement Response for Different Reinforcement Lengths (L)](image)

At footing displacements greater than \( \delta = 0.1B \) there is an obvious divergence of the six load-displacement responses in Figure 9.2. The unreinforced and nominally reinforced \( (L = 0.98B) \) responses, runs B40 and B40R1, are very similar, indicating that this smallest of reinforcement lengths is completely inadequate as it does not cover the critical part of the subgrade surface that is affected by the imposed interface stresses, i.e. the bearing
<table>
<thead>
<tr>
<th>Ref.</th>
<th>L</th>
<th>( P_f/s_u^{ps} )</th>
<th>( \delta_f )</th>
<th>( \beta )</th>
<th>Ref.</th>
<th>L</th>
<th>( P_f/s_u^{ps} )</th>
<th>( F_r/(B \ s_z^{ps}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>B40</td>
<td>0</td>
<td>8.33</td>
<td>-8.0°</td>
<td>35.9°</td>
<td>RB40</td>
<td>U</td>
<td>6.56</td>
<td>-</td>
</tr>
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<td>3.28</td>
</tr>
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<td>B40R</td>
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<td>-10.3°</td>
<td>40.9°</td>
<td>RB40R</td>
<td>R</td>
<td>10.49</td>
<td>3.28</td>
</tr>
</tbody>
</table>

Table 9.1: Run References for the Finite Element and Analytical Results Shown in Figure 9.2

The capacity of the clay, for both runs, is reduced because of the significant outward acting shear stresses that the clay sustains (illustrated in Figure 9.3). Furthermore, the length of reinforcement for run B40R1 is insufficient to build up shear stress.

For a reinforcement length of 1.47B, run B40R2, there is some distinct improvement in the load bearing capacity, compared to runs B40 and B40R1. This length of reinforcement is sufficient to pick-up the majority of the detrimental shear stresses at the interface and to exhibit something of a tensioned membrane effect, but it is not capable of acting as an effective tensioned membrane when it is forced to deform at footing displacements greater than \( \delta = 0.2B \). Whereas for the reinforcement lengths of \( L \geq 1.96B \), runs B40R3, B40R4 and B40R5, the tensioned membrane mechanism operates successfully causing the footing pressures to increase with displacement at a fairly constant rate. An additional relatively small amount of increased footing load is obtained for each of the longer reinforcement lengths up to the maximum limit given by B40R5 (equivalent to run B40R).

It is clear from the results presented that a large improvement in the maximum applied footing load, compared to the unreinforced case, can be obtained by including reinforcement at the base of the fill (≈ 30% for B40R5), even though the membrane does not cover the
entire base area of the fill as was previously thought to be necessary. This contradicts the theory that firm anchorage of the membrane is essential to obtain adequate reinforcement. Similarly, this would be the case for reinforced unpaved roads, provided the wheel load followed a defined path and did not wander beyond the membrane area.

The reinforcement length cannot be adjusted specifically in the Houlsby et al. (1989) method, however the reinforced analytical predictions (presented in Table 9.1) give reasonable approximations to the finite element results with increases in the maximum footing pressure due to the increase in the estimated load spread angle. Whereas, the unreinforced analytical value of $P_f$ is too low, because of the erroneous assumption of consistently outward acting shear stresses at the fill base, see Figure 9.3.

### 9.2.2 Normal and Shear Stress Distributions

Figure 9.3 shows the profiles of normal stress ($\sigma$) and shear stress ($\tau$) along the length of the reinforcement at the final footing displacement of $\delta = 0.6B$.

The upper normal stress distributions show a progressive increase in the magnitude of the normal stresses acting on the reinforcement for increasing reinforcement length. As greater lengths of reinforcement are introduced the magnitude of the normal force applied by the fill increases gradually from the minimum in run B40, to the maximum in run B40R5. The normal stress distribution for run B40R5 can be seen to coincide with that for run B40R by comparing Figures 9.3 and 6.12.

The upper normal stresses for run B40 start at $\sigma = 5.43 \, s_u^{ps}$ on the centre-line and after increasing only slightly diminish steadily to the residual self weight of the fill beyond about $x = 2.4B$. Whereas, for each increasing length of reinforcement, the centre-line normal stresses increase (up to a maximum of $\sigma = 6 \, s_u^{ps}$ for B40R5), and a greater peak normal stress is reached due to the tensioned membrane effect. The difference in magnitude between these upper normal stress distributions is an indication of the different amounts of tensioned membrane effect acting in each reinforced two-layer soil system.

9-5
NORMAL STRESS

upper

\[ \frac{\sigma}{s_u^{PS}} \]

\[ x / B \]

lower

\[ \frac{\sigma}{s_u^{PS}} = 34^\circ - 44^\circ \]

\[ x / B \]

SHEAR STRESS

upper

\[ \frac{\tau}{s_u^{PS}} \]

\[ x / B \]

lower

\[ \frac{\tau}{s_u^{PS}} \]

\[ x / B \]

Figure 9.3: Normal Stress \( (\sigma/s_u^{PS}) \) and Shear Stress \( (\tau/s_u^{PS}) \) along Interface for Varying Reinforcement Lengths \( (L) \), (at \( \delta = 0.6 \times B \))

The erratic nature of the lower normal stress profiles for runs B40R1 \( (L = 0.98 \times B) \) and B40R2 \( (L = 1.47 \times B) \) is caused by the abrupt ending of the reinforcement. The lower normal stress distributions are influenced by the outward acting shear stresses on the subgrade surface and for the unreinforced, B40, and nominally reinforced, B40R1, analyses especially, the normal stresses are noticeably reduced, at distances away from the centre-line, in comparison to the other runs. The runs B40R5, B40R4, B40R3 and to a degree B40R2,
approximately exhibit the theoretical uniform normal stress distribution with \( \sigma = 5.9 s_u^{P_S} \), as estimated by equation (6.3) where \( N_e = 1 + 3 \pi / 2 \) due to the inward acting shear stresses on the subgrade (Mode 2 in Figure 6.16).

From the plot of the lower normal stress distributions the small effect of an improved load spread is visible for the runs using reinforcement of an effective length (runs B40R5, B40R4 and B40R3), compared to those using an inadequate reinforcement length (runs B40, B40R1 and B40R2). The actual calculated load spread angles (\( \beta \)) are given in Table 9.1. However, the minor improvement in \( \beta \) does not contribute to the reinforcing effect as substantially as the tensioned membrane mechanism, which is a similar situation to that found during the investigation of the influence of increasing reinforcement stiffness (Section 8.2.2).

As \( L \) increases the pattern of the upper shear stress distributions along the length of the interface progressively changes as it converges to that of the adequately reinforced case, run B40R5. Directly under the footing the inward acting shear stresses increase from low stresses in run B40 (\( \tau_{\text{peak}} = -0.7 s_u^{P_S} \)) to high stresses in run B40R5 (\( \tau_{\text{peak}} = -1.4 s_u^{P_S} \)), and beyond \( x = 1 \) \( B \) they increase from comparatively small to much greater outward shear stresses. The horizontal distance at which the reinforcement ends in runs B40R1 and B40R2 is clearly defined by the sudden reduction in the outward shear stresses.

The distributions of shear stress on the lower reinforcement surface are rather unclear, but the trend is that for the unreinforced run B40 and nominally reinforced runs B40R1 and B40R2 the stresses transpose from inward to large outward acting (\( \tau_{\text{peak}} = s_u^{P_S} \)) as \( x \) increases (which reduces the load bearing capacity as seen in Figures 9.2 and 9.3). While, for the more fully reinforced runs B40R3, B40R4 and B40R5, the distributions of shear stress almost entirely remain inward acting as \( x \) increases, except beyond the end point of the reinforcement in run B40R3 (\( x = L = 1.96 B \)). The spurious peak inward acting shear stresses for runs B40R2 and B40R3, which exceed the shear strength of the clay, coincide with the end points of the different reinforcement lengths and are caused by discretization errors (see Section 5.4.5).
For this particular combination of two-layer soil properties and dimensions it is suggested that a reinforcement length of $L = 2B$ is sufficient to provide adequate reinforcement, since the tensioned membrane and shear stress mechanisms operate effectively. Note that there are still some diminishing shear stresses at the interface beyond $x = 2B$, which have an effect on the induced reinforcement force, see Figure 9.4.

9.2.3 Reinforcement Tension Distribution

Figure 9.4 shows that as the reinforcement length ($L$) increases, the force in the reinforcement ($F_r$) also increases throughout the membrane length, up to a maximum limit given by run B40R5 ($L = 4B$).

\[
\frac{F_r}{B \cdot s_{ud}^{ps}}
\]

\[
\frac{x}{B}
\]

**Figure 9.4: Reinforcement Tension ($F_r/(B \cdot s_{ud}^{ps})$) for Different Reinforcement Lengths ($L$), (at $\delta = 0.6B$)**

The reinforcement tensions obtained for runs B40R5 and B40R4 are virtually identical, but for run B40R3 ($L = 1.96B$) the tension is slightly smaller (10% lower at the peak value, where $x = 1B$), because $L$ is not quite long enough to completely pick-up all of the force applied by the shear stresses at the interface beyond $x = 2B$. As a consequence of the even shorter reinforcement lengths in runs B40R2 and B40R1 their tension profiles are reduced still further. Clearly the smaller the reinforcement force, the less contribution the tensioned membrane mechanism can provide in reinforcing the system.
The analytical values of the reinforcement tension, Table 9.1, vary from the peak finite element results, Figure 9.4, because the clay shear forces are neglected. However, successively larger values of $F$, are predicted due to the nominal increases in both the load spread angle and the outward acting footing frictional force, except for run RB40R5 which has a slightly lower value of $\beta$.

9.2.4 Slip and Deformation along Soil-Reinforcement Interface

The plots in Figure 9.5 of the upper and lower relative slip, $u_r$, for each reinforcement length are a little confused, because of the erratic oscillations in $u_r$ at the reinforcement ends. As $L$ increases, the magnitude of $u_r$ along the upper reinforcement surface tends to increase between $x = 1B$ and $3B$, in a manner similar to the upper shear stress distributions. Furthermore, the lower surface $u_r$ values change from positive to negative slip, between $x = 1.8B$ and $4B$, for reinforcement lengths of $L \geq 1.96B$, which generates the inward clay shear stresses shown in Figure 9.3.

The profiles of the deformed membranes and the unreinforced interface, shown in Figure 9.6, illustrate that initial increases in the reinforcement length reduce the interface displacements appreciably, but that once the reinforcement reaches a length of $\approx 2B$ any continued increase in $L$ has no significant influence.

9.3 General Conclusions

In practice a geosynthetic material at the interface of a two-layer soil system will often provide other functions, in addition to reinforcement, such as separation or filtration, which would clearly necessitate that the membrane covered the entire interface area. However, as a reinforcing layer exclusively, this parametric study shows that the width of the membrane needs only to be sufficiently wide enough to cover the area of subgrade influenced by both the normal and shear stresses generated at the interface. These fill and clay shear stresses exert a force on the membrane which produces the reinforcement tension, the magnitude of which is determined by the proportion of stresses that the membrane is able to pick-up.

9-9
Figure 9.5: Relative Slip \( u_r \) along Interfaces

Figure 9.6: Final Deformed Shape of the Membranes

Therefore, with a reinforcement of width just greater than the imposed loaded area the tensioned membrane effect and the shear stress mechanism can operate adequately and provide a completely reinforced system. The significance of this is that the reinforcement...
does not need to be anchored in any way prior to placing the fill, as was previously thought essential, to achieve the tensioned membrane effect. Thus, effective reinforcement can be achieved with only a minimum membrane width.

This result has further significance for the development of analytical models of membrane behaviour in two-layer soil systems. Many authors have attempted to develop simple analytical models to predict the membrane behaviour, but these, in general, have required fixities to be assumed at various points outside of the loaded area. In the development of any new models of behaviour it is clearly necessary to avoid the assumption of these spurious fixities.
CHAPTER 10:

INFLUENCE OF CLAY STRENGTH

10.1 Introduction

Reinforcement is often included in two-layer soil systems if the ground conditions are of poor quality with low, or seasonally very changeable soil strengths. The undrained shear strength of the subgrade soil, \( s_u \), is therefore an important design parameter.

10.2 Parametric Study

For this study of how variations in the undrained shear strength of the clay effect the overall performance of a reinforced and unreinforced two-layer soil system, the parametric study is conducted for both plane strain and axisymmetric conditions, using meshes B and AXB (Figures 5.2 and 6.18) and the dimensions and properties of the soil and membrane given in Appendix 6A.

The plane strain shear strength values, \( s_u^{PS} \), selected for this finite element parametric study are representative of the range of clay strengths that are typically encountered on the sites of two-layer soil systems; 10 kPa, 30 kPa and 70 kPa. These undrained shear strengths are classified in the British Standard CP 2004 (1972) as "very soft", "soft" and "firm", respectively.

Although \( s_u^{PS} \) is varied in this parametric study, the clay shear modulus is kept constant at \( G_c = 1600 \text{ kPa} \) and similarly the undrained Young’s modulus, \( E_u \), which is given by:

\[
E_u = 2 \ G_c \ (1 + \nu_c)
\]  

\[ 10.1 \]
where the Poisson's ratio is $v = 0.49$.

The von Mises constitutive model used in OXFEM (see Chapter 3) requires the triaxial compression value of shear strength, $s_u^T$, which is related to the plane strain value, in the limit that the elastic strain rates are negligible in comparison with the plastic strain rates, by the equation given in Burd (1986):

$$s_u^{PS} = \frac{2}{\sqrt{3}} s_u^T \quad (10.2)$$

The plane strain value of the clay shear strength is used to categorise each finite element run for both the plane strain and axisymmetric analyses.

10.2.1 Load-Displacement Response

The footing pressure-displacement responses for each of the six plane strain and six axisymmetric, large displacement, finite element runs are illustrated in Figures 10.1 a) and b), and the results at a final displacement of $\delta = 0.6 \, B \, R$ are presented in Table 10.1, along with the comparative analytical predictions from the Houlsby et al. (1989) and Houlsby and Jewell (1990) methods. Note that the footing pressure, $P$, is normalised with respect to the clay shear modulus, $G_c$, rather than the clay shear strength as used in previous analyses.

The plane strain loading responses shown in Figure 10.1 a) for each value of $s_u^{PS}$, regardless of whether reinforced or unreinforced, possess an identical elastic stiffness. Similarly, the six axisymmetric loading responses shown in Figure 10.1 b) each have an identical elastic stiffness, which is approximately twice that of the plane strain. Clearly, the final footing pressures for the axisymmetric analyses are greater than the those for the corresponding plane strain analyses, Table 10.1.

The larger the prescribed value of $s_u^{PS}$, the greater the amount of footing displacement required before the plane strain and axisymmetric two-layer soil systems begin to yield plastically. At the same point of deviation from the linear elastic slope, the effect of the reinforcement becomes evident by maintaining a higher loading capacity compared to the equivalent unreinforced analysis. However, the proportional amount of loading
Figure 10.1: Pressure-Displacement Responses for Different Clay Strengths ($s_u^{PS}$)

Improvement obtained by including reinforcement reduces as the clay shear strength increases for both plane strain and axisymmetric conditions, i.e. for plane strain there is a 50% improvement for $s_u^{PS} = 10 \, kPa$, 31% improvement for $s_u^{PS} = 30 \, kPa$ and 18% improvement for $s_u^{PS} = 70 \, kPa$ at a footing displacement of $0.6 \, B$, while for axisymmetry there is a 79% improvement for $s_u^{PS} = 10 \, kPa$, 52% improvement for $s_u^{PS} = 30 \, kPa$ and 31% improvement for $s_u^{PS} = 70 \, kPa$ at a footing displacement of $0.6 \, R$. 

10-3
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<th>$P_f/G_c$</th>
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**Table 10.1: Run References for the Plane Strain and Axisymmetric Finite Element and Analytical Results Shown in Figures 10.1**

From Table 10.1 it is clear that as $\delta_u^{ps}$ increases the outward acting angle of friction on the footing base ($\delta_f$) decreases for both the plane strain and axisymmetric analyses, although the actual footing frictional force ($P_f B \tan \delta_f$, or $P_f R \tan \delta_f$) increases.

Similarly the load spread angle ($\beta$) also decreases with increasing clay shear strength, as seen clearly in the normal and shear stress distributions in Figures 10.2 and 10.3. The
large values of $\beta$ for the very soft clay analyses are comparable to those found by Milligan et al. (1989). By comparing the results of the large-scale experimental axisymmetric tests of Fannin (1986) and the Houlsby and Jewell (1990) analytical method, Milligan et al. (1989) found that an appropriate value of $\beta$ was in the range 45° - 50°, for the case of a strong fill ($\phi = 45^\circ$) over weak clay ($s_u = 8$ kPa). The smaller range of $\beta$ values for the axisymmetric analyses, compared to plane strain, is due to the more rapid reduction of the vertical stress within the fill than occurs in plane strain conditions.

The reinforced ultimate footing loads predicted by the Houlsby et al. (1989) and Houlsby and Jewell (1990) methods for the plane strain and axisymmetric cases, Table 10.1, are mostly reasonable approximations to the finite element results. For the unreinforced cases, however, the analytical solutions are generally less comparable to the finite element results, because of the incorrect assumption that the shear stresses are consistently outward acting at the fill base, see Figures 10.2 and 10.3.

10.2.2 Normal and Shear Stress Distributions

All the normal ($\sigma$) and shear ($\tau$) stress distributions shown in Figures 10.2, for plane strain, and 10.3, for axisymmetry, are normalised with respect to the particular value of the plane strain shear strength prescribed for that individual run.

By increasing the shear strength from $s_u^{PS} = 10$ kPa to 70 kPa there is a noticeable reduction in the non-dimensionalised normal stresses (although an increase in the absolute values), in both the reinforced and unreinforced plane strain and axisymmetric analyses. This is partly because the magnitude of the inward shear stresses acting on the clay, just outside the load spread area, reduce from $\tau = -1 s_u^{PS}$ (slightly greater than the shear strength of the clay due to discretization errors) to about zero as the clay shear strength is increased and consequently the bearing capacity factor of the clay decreases. For plane strain conditions this reduction in $N_c$ is from the absolute maximum of $(1 + 3 \pi / 2)$ to $(2 + \pi)$ (i.e. from Mode 2 to Mode 1 in Figure 6.16), whereas for axisymmetric conditions the approximate reduction in $N_c$ is from $\approx 1.11 \times 6.3$ to 6.3 (see Section 6.4.2). Thus, for the $s_u^{PS} = 70$ kPa analysis the lower normal stresses at the footing centre-line approximately reduce to the plasticity solution of $(\pi + 2) s_u^{PS} + \gamma_p D_f \approx 5.2 s_u^{PS}$ in plane strain and
6.3 (\sqrt{3}/2) s_u^{PS} + \gamma D \approx 5.5 s_u^{PS} in axisymmetry. Additionally, an increase in shear strength also decreases the load spread angle (see Table 10.1), which is evident in all of the upper and lower normal stress distributions illustrated in Figures 10.2 and 10.3. This same effect was experimentally observed by Love (1984), (as shown in Tables 7.3 and 7.4), and is also predicted by the theoretical relationship derived from the Meyerhof (1974) analysis, given as equation (7.6).

The difference between the upper and lower reinforced normal stress profiles gives a measure of the tensioned membrane effect, which clearly has a diminishing influence as the clay strength increases in both the plane strain and axisymmetric analyses. This result is also apparent from the smaller percentages of improvement obtained in the pressure-displacement responses as \( s_u^{PS} \) increases, Figure 10.1. By comparing the plane strain and axisymmetric upper normal stress distributions it is clear that the tensioned membrane effect is greater in axisymmetric conditions for any particular value of \( s_u^{PS} \).

The magnitudes of the plane strain and axisymmetric non-dimensionalised shear stresses on the upper and lower reinforcement surface decrease (but increase in absolute value) for increasing \( s_u^{PS} \), and effect a significantly reduced area along the reinforced interface due to the smaller load spread angle. The shear stress mechanism of the reinforcement is evident by comparing the consistently outward acting shear stresses on the lower reinforcement surface to the alternating inward/outward acting shear stresses at the unreinforced interface. The exceptionally high peak inward acting shear stresses for runs B40R10 and AXB40R10 are spuriously caused by discretization errors, as explained in Section 5.4.5. The peak outward shear stresses in the unreinforced analyses reach a critical value of 1 \( s_u^{PS} \) at around \( x = 1.5 \, B \rightarrow 2 \, B \) and \( r = 1.5 \, R \rightarrow 1.8 \, R \).
Figure 10.2: Normal ($\sigma / s_u^{ps}$) and Shear ($\tau / s_u^{ps}$) Stresses along Interface for Varying Clay Strengths ($s_u^{ps}$) in Plane Strain, (at $\delta = 0.6 \, B$)
**Axisymmetry**

**NORMAL STRESS**

*upper interface*

![Graph showing normal stress ratio for reinforced conditions.](image1)

*lower interface*

![Graph showing normal stress ratio for reinforced conditions.](image2)

*unreinforced interface*

![Graph showing normal stress ratio for unreinforced conditions.](image3)

**SHEAR STRESS**

*upper interface*

![Graph showing shear stress ratio for reinforced conditions.](image4)

*lower interface*

![Graph showing shear stress ratio for reinforced conditions.](image5)

*unreinforced interface*

![Graph showing shear stress ratio for unreinforced conditions.](image6)

**Figure 10.3**: Normal \((\sigma / S_u^{ps})\) and Shear \((\tau / S_u^{ps})\) Stresses along Interface for Varying Clay Strengths \((S_u^{ps})\) in Axisymmetry, (at \(\delta = 0.6R\))
10.2.3 Reinforcement Tension Distribution

The reinforcement tensions (longitudinal, \( F_r \), in plane strain and tangential, \( F_{\text{tang}} \), and circumferential, \( F_{\text{circ}} \), in axisymmetry) developed at the final footing displacement of \( \delta = 0.6 \, B \); \( R \) are normalised in Figure 10.4 by the product of \( B \), or \( R \), and the clay shear modulus, \( G_c \), rather than \( B \, s_u^{PS} \) as used in the previous parametric studies.

The peak tensions, reached at about \( x = 1 \, B \) in plane strain and \( r = 0.7 \, R \) in axisymmetry, increase with higher clay strengths because of the larger absolute shear forces acting on the membrane. However, the rate of reduction of the longitudinal and tangential reinforcement tensions along the membrane is quicker, in terms of distance, for higher values of \( s_u^{PS} \), because of the smaller load spread area in which the upper and lower shear stresses are concentrated, see Figures 10.2 and 10.3. The peak circumferential tensions are slightly lower than the peak tangential tensions, for any particular run, but do not reduce to zero as quickly and extend for a greater horizontal distance.

The analytical predictions of the tensions, given in Table 10.1, are poor approximations of the peak finite element values. For \( s_u^{PS} = 10 \, kPa \) and \( 30 \, kPa \) the analytical values of the tensions are too low and for \( s_u^{PS} = 70 \, kPa \) they are too high, for plane strain and axisymmetry.

10.2.4 Slip and Deformation along Interfaces

The relative slip, \( u_0 \), along the unreinforced and the upper and lower reinforced surfaces is illustrated in Figure 10.5, for plane strain and axisymmetry. Significant slip continues for greater horizontal distances in the plane strain analyses than exists in axisymmetry.

As the clay strength increases, the magnitude of the slip at the unreinforced and upper reinforced surfaces increases also, in both plane strain and axisymmetry, as do the absolute shear stresses shown in Figures 10.2 and 10.3. Along the lower reinforced surface, however, the magnitude of the slip actually decreases as \( s_u^{PS} \) is increased, in contrast to the magnitude of the lower absolute shear stresses. It is noteworthy that for the unreinforced very soft clay analyses (\( s_u^{PS} = 10 \, kPa \)), the general trend of the fill slipping out over the clay, diminishes to approximately zero in axisymmetry, while for plane strain the slip actually reverses so that the clay tends to slide out from under the fill.
Figure 10.4: Reinforcement Tensions for Different Clay Strengths, (at $\delta = 0.6 \frac{B}{R}$)
Figure 10.5: Relative Slip \( (U_r) \) along the Interfaces

The shape of the final deformed interfaces, shown in Figure 10.6, vary according to the clay strength, with smaller vertical displacements for the stiffer clay. The reinforcement tends to reduce the interface deformations, although for plane strain conditions the magnitude of the vertical displacements, beyond \( x = 2B \), is greater generally than those for axisymmetry beyond \( r = 2R \).

10.3 General Conclusions

There is a dramatic effect upon the mechanics of a plane strain and an axisymmetric two-layer soil system for variations in the undrained shear strength of the clay subgrade. Increasing the clay shear strength obviously necessitates an increased footing pressure to obtain the
same vertical displacement, but simultaneously this reduces the proportional contribution of the tensioned membrane mechanism to the overall reinforcement of the system. Additionally, the load spread angle also reduces with increasing clay strength, so that the area along the interface in which the normal and shear stresses act, diminishes significantly.

The Houlsby et al. (1989) and Houlsby and Jewell (1990) design methods give reasonable indications of the value of the ultimate footing load on a reinforced two-layer soil system for a variety of clay strengths, but less accurate estimates for the unreinforced system. The predicted reinforcement tensions are also inaccurate, because of the simplified approach used in the limit equilibrium methods to estimate the complex arrangement of inward and outward acting shear forces applied to the membrane.

**Figure 10.6: Final Deformed Shape of the Interfaces**
CHAPTER 11:

ANALYSIS OF LOW FRICTION MEMBRANES
IN GRANULAR BASES

11.1 Introduction

The application of the newly developed axisymmetric interface element (detailed in Chapter 5) to the analysis of the use of membranes in granular base foundations is described in this chapter. The particular application considered is for the comparatively new and relatively unresearched problem of low friction polymeric membranes incorporated within the compacted fill base of oil storage tanks, to provide secondary containment against oil leakage from damaged, or corroded tanks. The purpose of this study is to investigate numerically how the presence and configuration of a low friction membrane influences both the surface displacements and the soil deformations at the tank edge, for various values of plane strain fill friction angle ($\phi_{ps}$) and shear modulus ($G_f$).

Extensive research has been carried out at the University of Oxford on the use of low friction membranes in oil tank bases and the problem of predicting the amount of edge displacement developed by filling the tank, i.e. 'edge cutting'. This is important because the rut created by edge cutting can become a moisture trap, leading to accelerated tank corrosion. This chapter gives a qualitative synopsis of that research, which is fully documented in Burd et al. (1993). However, due to contractual agreements on confidentiality, the complete set of data and detailed analyses have been omitted.
11.2 Design of Oil Storage Tanks

Large steel oil storage tanks are flexible structures and transmit the weight of their liquid contents to their foundation as a uniform distributed load. Designs for such tanks have to accurately predict and allow for deformations of the structure and the underlying ground, which clearly requires a rigorous site investigation and precise design specifications. Unfortunately this has not always been achieved and full-scale failures of oil tanks have not been uncommon in the past, e.g. Nixon (1949) and Brown and Paterson (1964).

If the subgrade soil is strong enough to support the weight of the tank, the simplest form of foundation consists of a pad of granular material, usually less than 1 m thick, placed directly on the exposed subgrade, Figure 11.1. The pad is generally made thicker at the centre than at the edges (typically a 1:120 slope) since central settlements are expected to be greater than the edge settlements. The dimensions of a typical oil storage tank, as assumed for this study, and the base edge details are shown in Figure 11.1.

![Figure 11.1: Simple Oil Storage Tank Foundation](Schematic Diagram)

The risk of ground pollution and the structural dangers of reduced foundation stability, caused by any amount of oil leakage, have become of increasing concern recently. This has led to the use of impermeable membranes within the fill layer for leak detection and containment purposes.
The recommended Standard for welded steel oil storage tanks (A.P.I. 650) illustrates typical designs for the installation of membranes as undertank leak detection systems, for example Figure 11.2.

![Figure 11.2: Typical Tank Perimeter Design](image)

Figure 11.2: Typical Tank Perimeter Design
(After A.P.I. 650 Appendix I)

It is possible, however, that the inclusion of a low friction membrane may introduce a plane of weakness, which could lead to unacceptably increased settlements under the tank. Additionally, the presence of a membrane beneath the tank edge may exacerbate the problem of soil heave around the perimeter of the tank caused by edge cutting.

11.3 Idealization of the Problem

The problem is idealized in order to conduct a large displacement axisymmetric finite element investigation. Since it is the local edge deformations within the fill that are important in this study, it is not necessary to include the layer of subgrade soil in the analysis. This means that the fill deformations are computed correctly, but that the general elastic settlements, predicted beneath the tank as a whole, would be inexact with this approach. Additionally, the finite element mesh only needs to extend a sufficient distance, either side of the edge, so that any mesh boundary effects are removed (see Figure 11.5).
The fill is represented as an elastic perfectly-frictional material, modelled by the Matsuoka yield function (discussed in chapter 3), and the steel tank is modelled using a linearly elastic perfectly-plastic model (based on the von Mises yield criterion). The soil and tank are both modelled using fifteen-noded axisymmetric triangular continuum elements (illustrated in Figure 3.1) with a sixteen-point Gauss quadrature. Some initial analyses employed a thirteen-point Gauss quadrature, but this proved to be numerically unstable. The membrane is modelled using the ten-noded axisymmetric frictional interface element with a seven-point Gauss rule, as described in Chapter 5.

The shell plate of the tank is modelled by a 'stub wall' of thickness $0.025\, m$ and height $1.5\, m$, rather than the complete $20\, m$ height which would require an excessively large number of elements. It is thought, however, that the important features of the interaction between the tank and the foundation are modelled adequately by this approach. A horizontal constraint is applied to the inside bottom edge of the annular plate in order to simulate the presence of the sketch plate. The simplified tank base details, used in the finite element investigation, are shown in Figure 11.3.

![Figure 11.3: Idealized Model for Tank Edge](image)

*(After Burd et al. (1993))
Due to the large range of material stiffnesses considered in this problem (i.e. the relatively high steel stiffness compared to the low soil stiffness), ill-conditioning of the finite element global stiffness matrix might be expected to occur. Such ill-conditioning can be moderated by reducing, if possible, the high steel stiffness. Therefore, the prescribed stiffness of the annular base plate is reduced by a factor of 125 and the thickness is increased by a factor of 5, to preserve the important bending stiffness, (as shown in Figure 11.3). Similar modifications of the tank wall properties, however, are considered inappropriate since both the bending and axial stiffnesses of the wall have an important effect on its structural behaviour.

The finite element calculations are all load controlled, with the loads taken to be the values typical of an oil tank. The stresses due to the fill self weight are assigned initially, followed by two separate loading stages. The first stage applies increments of nodal forces equivalent to the combined self weights of the tank (41 kN/m) and the oil (10 kN/m$^3$). The tank self weight is applied at the base of the wall. The original intention was not to apply these loads simultaneously, but as two separate stages. However, this proved difficult because of the numerical instability caused by the eccentric loading of the annular plate under the tank self weight only, whereas by combining it with the oil weight the annular plate is loaded more evenly. The second stage consists of the further application of load, at the base of the wall, corresponding to both the wind (15 kN/m) and the roof surcharge (5 kN/m) loading.

Hydrostatic loads are calculated from the hydrostatic stresses for increments of the oil depth $H$, shown in Figure 11.4, and applied at the relevant nodes. This calculation is based on the standard virtual work approach and the vector of the nodal forces, $\mathbf{P}$, is given by:

$$
\mathbf{P} = 2\pi \int_E [N]^T \mathbf{w} r \, ds \tag{11.1}
$$

where the shape function matrix, $[N]$, is given by the equations (4.64) to (4.69) and $w$ is the hydrostatic pressure. The integration is performed over the element length, using a
seven-point Gaussian quadrature scheme. The oil depth is increased to a maximum of 20 m. For every increment of oil level inside the tank, one of three loading cases exists for each of the elements along the base and inside wall of the tank:

1) All the nodes are above the oil surface:
   No nodal loads are applied.

2) The oil surface intersects the element:
   Only relevant for vertical elements, where the submerged nodes are loaded linearly with height and those above the oil level are allocated zero nodal force.

3) All the nodes are below the oil surface:
   For horizontal elements the pressure is constant,
   For vertical elements the pressure over the element is linear with height.

![Figure 11.4: Application of Oil Loading](image)

(After Burd et al. (1993))

This study considers the oil tank foundation initially without a membrane (Mesh I, Figure 11.5) and then three cases for different configurations of the membrane position within the fill. Firstly, with the membrane horizontal across the entire mesh width, at a depth of \( d = B / 3 \), where \( B \) is the width of the annular plate equal to 775 mm (Mesh II, Figure 11.5). Secondly, for a similar arrangement, but with \( d = 2 B / 3 \) (Mesh III, Figure 11.6) and finally with the membrane again horizontal at a depth of \( d = 2B / 3 \) beneath the tank, but inclined at 50° under the annular plate and terminating at the outside edge of the
annular plate (Mesh IV, Figure 11.7). The 50° inclination, from the horizontal, of the membrane in mesh IV is considered to represent the worst recommended design case (Figure 11.2). The end detailing of the membrane in Mesh IV requires that the last upper node of the interface element (i.e. node 7 in Figure 5.16) is common with the annular plate edge and that the last lower interface node (i.e. node 2 in Figure 5.16) is common with the underlying soil.

Figure 11.5: Mesh I - No Membrane
(Mesh II - Membrane at level indicated)
(After Burd et al. (1993))
Figure 11.6: Mesh III
(After Burd et al. (1993))

Figure 11.7: Detail of Mesh IV with Element Numbers
(After Burd et al. (1993))
11.4 Specification of Material Properties

A series of finite element calculations is described, for each of the different meshes, with various prescribed values of plane strain fill friction angle ($\phi_{PS}$) and shear modulus ($G_f$).

Three different values of $\phi_{PS}$ are chosen (32°, 40° and 48°) which represent a material with a critical state friction angle $\phi_{cv}$ of 32° compacted to different densities. The Poisson’s ratio of the fill, $\nu_f$, is set to 0.2 and the value of the plane strain dilation angle, $\psi_{PS}$, is assumed to be given by the empirical relationship proposed by Bolton (1986):-

$$\phi_{PS} = \phi_{cv} + 0.8 \psi_{PS} \quad (11.2)$$

The variations in the plane strain dilation angle are believed to have only a minor influence on the behaviour of the fill for this type of loading situation, as suggested in Section 7.2, but nevertheless are considered here for the sake of completeness.

Two values of soil shear modulus are selected, $G_f = 3\,000\, kPa$ as a lower bound stiffness and $G_f = 12\,000\, kPa$ as an upper bound value. The finite element calculations are listed in Table 11.1, referenced by the relevant mesh used (i.e. Mesh I, II, III, or IV), along with the corresponding final vertical displacements, calculated at the outside edge of the annular plate for each run.

For runs II, III and IV the membrane is assumed to have an angle of friction $\phi_i$ of 10°, which is a credible lower bound value for smooth membranes, and zero dilation angle. It is important to choose appropriate values for the elastic shear and normal stiffnesses ($k_s$ and $k_n$ respectively), because values that are too high will result in ill-conditioning and too low will lead to significant additional shear and normal displacements. The stiffnesses are derived from the equations:-

$$k_s = \frac{G_f}{L_c} \quad (11.3)$$

$$k_n = \frac{E}{L_c} = \frac{2\,G_f(1 + \nu_f)}{L_c} \quad (11.4)$$

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where the characteristic length $L_c$ corresponds to a nominal thickness of the interface. It is assumed that $L_c = 0.1 \ m$ for the calculations using Meshes II and III, and $0.05 \ m$ for the Mesh IV runs. This increased stiffness for the sloping membrane runs was considered appropriate in order to assure numerical stability.

<table>
<thead>
<tr>
<th>Runs Referenced by Mesh No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_{PS}$</td>
</tr>
<tr>
<td>32°</td>
</tr>
<tr>
<td>40°</td>
</tr>
<tr>
<td>48°</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Corresponding Vertical Displacements of Annular Plate Perimeter (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_{PS}$</td>
</tr>
<tr>
<td>32°</td>
</tr>
<tr>
<td>40°</td>
</tr>
<tr>
<td>48°</td>
</tr>
</tbody>
</table>

Table 11.1: Finite Element Referencing and Calculated Edge Displacements

In all the analyses the bulk unit weight of the fill ($\gamma_f$) is taken to be $18 \ kN/m^3$. The initial vertical stresses ($\sigma_v$) are computed directly from the self weight and the horizontal stresses for runs I, II and III, are calculated as $K_0 \sigma_v$, using a specified value of the earth pressure coefficient, $K_0$, given by the Jaky expression:

$$K_0 = 1 - \sin\phi_{PS}$$  \hspace{1cm} (11.5)

For runs IV this value of $K_0$ is inappropriate, since the mobilised friction angle on the sloping portion of the membrane would exceed 10°. It is possible to show that the value of
$K_0$, which would just cause failure, is given by:

\[ K_0 = \frac{1 - \sin \phi_i}{1 + \sin \phi_i} \approx 0.7041 \] (11.6)

Therefore, a value of $K_0 = 0.71$ is used for the Mesh IV runs.

### 11.5 Discussion of Results

The results presented in Table 11.1 show that as the fill strength increases all of the predicted edge vertical displacements reduce significantly, and that increasing the shear modulus from 3 000 kPa to 12 000 kPa has the effect of decreasing the displacements by a factor of approximately 3.5.

Incorporating a low frictional membrane into the fill, clearly increases the edge displacements of the annular plate, compared to the case where the membrane is absent (Mesh I). For the configurations of the membrane that extend horizontally beyond the tank perimeter (Meshes II and III), the additional vertical displacements are all relatively small, with no obvious indication as to which of the membrane depths is most adverse. The sloping membrane case (Mesh IV), however, produces the largest edge displacements (except for $\phi_{ps} = 40^\circ$ and $G_f = 3000$ kPa, run IV3, which is thought to be spuriously low).

The edge cutting of the annular plate into the fill and the exact relative distortions within the soil are seen clearly in the final deformed meshes, two of which are illustrated in Figures 11.8 and 11.9. Small rotations of the loaded annular plate are discernible, particularly for run IV1 (Figure 11.9). Inside of the annular plate the vertical settlements are approximately uniform and these diminish with depth, while the horizontal movements are comparatively insignificant. The relative soil slip between the upper and lower membrane surfaces is visible from these deformed meshes (investigated in more detail later) and, in Figure 11.9 particularly, the interpenetration of the continuum elements across the slip plane is also perceptible. This small overlap of the elements is due to the finite normal stiffness ($k_n$), which is assigned to the interface elements to avoid numerical instability, and is approximately equal to $\sigma_n/k_n$. 

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Figure 11.8: Undeformed and Corresponding Deformed Mesh for run II5
(After Burd et al. (1993))

Figure 11.9: Deformed Mesh for run IV1 (After Burd et al. (1993))
The performance of the membrane in the analysis is assessed by studying the interface element Gauss point stresses at the end of run II5. The plot of the variation of normal stress along the membrane length (Figure 11.10) shows that $\sigma$ is constant under the tank at about 200 kPa, due to the oil loading, and equal to the fill self weight outside of the annular plate. Directly under the tank wall $\sigma$ reaches a peak of approximately 350 kPa, due to the extra imposed tank loading, and at the inside edge of the annular plate there is a stress relief, reducing $\sigma$ to only 150 kPa, caused by the outwards rotation of the annular plate.

Figure 11.11 shows the mobilised angle of friction on the membrane ($\phi_i'$), which is calculated from equation (11.7):

$$\phi_i' = \tan^{-1}\left(\frac{\tau}{\sigma}\right)$$  \hspace{1cm} (11.7)

where $\tau$ is the shear stress at the interface, (inwards acting shear is assumed positive, for Figure 11.11). For a distance of about 2 m beyond the outside edge of the annular plate the full friction angle of 10° is mobilised. This is followed by a sudden drop and sign change in the mobilised angle, caused by the reversal in direction of the shear stresses from outward to inward acting. This signifies that the soil below the membrane is being squeezed outwards from under the upper layer. Directly beneath the annular plate there is some mobilisation of friction, but insufficient to cause slipping. This is reflected in Figure 11.12, which shows the relative movement of the soil above the membrane compared to that below ($u_r$). The largest relative slipping, reaching a maximum of about 12 mm, is within 1 m of the tank wall centre-line, although the full friction is mobilised for a distance of up to 2 m. This is because beyond 1 m the normal stress is already very small and the nominal relative slip that occurs (i.e. < 1 mm) is enough to mobilise the full friction. The small relative movements under the annular plate are due to the finite shear stiffness of the interface elements ($k_s$) and are approximately equal to $\tau / k_s$. 
Figure 11.10: Normal Stress ($\sigma$) on Membrane

Figure 11.11: Mobilised Friction Angle ($\phi'$) on Membrane

Figure 11.12: Relative Slip ($u_r$) along Membrane
11.6 Conclusions

This study demonstrates that the newly developed axisymmetric interface element formulation performs properly. It also provides a useful insight into the way low friction membranes influence the behaviour of granular bases for large oil storage tanks.

A sloping ended membrane is found to induce larger relative displacements (≈ 37% maximum increase, for case considered) at the annular plate edge, than occur for a horizontally placed continuous membrane (≈ 13% maximum increase), in comparison to the displacements which occur when no membrane is used.

The important conclusion to be drawn from this study is that the influence of the membrane is relatively small compared to the influence of the soil friction angle. By incorporating a membrane of friction angle 10°, the relative additional displacements incurred are equivalent to decreasing the peak fill friction angle by about 3° to 5°.

It is suggested that the returns obtained, in terms of safety and environmental protection, from incorporating a low friction impermeable membrane as a secondary containment within the base of a newly constructed oil tank, far outweigh the additional material costs involved.
CHAPTER 12:

CONCLUDING REMARKS

The large displacement, large strain, finite element code OXFEM was modified, as part of the research described in this thesis, to include both plane strain and axisymmetric elastic-perfectly frictional interface elements, as well as axisymmetric linear elastic membrane elements. These new formulations are capable of modelling large global displacements and finite rotations, although relative displacements within the interface element are constrained to being infinitesimal. The finite element equations are based on an Updated Lagrangian description of deformation.

The advantages of using interface elements are that, firstly they allow the properties of the soil-reinforcement interface to be varied independently of the soil properties and secondly they provide a convenient way of extracting the stresses acting at the soil-reinforcement boundary. Although the interface element material stiffness has been limited, in the calculations described in this thesis, to using only the purely elastic case for the sake of numerical stability, the stress updating procedure takes full account of plasticity ensuring that the overall solution for the stresses is correct.

The axisymmetric membrane formulation is rigorously verified through a series of closed form elastic test problems, with the elements placed at various inclinations. Similarly, the axisymmetric interface formulation is tested, but the difficulty of finding appropriate closed form test analyses has meant that the validation is less comprehensive for this element. For the tests undertaken, the finite element results show that the new formulations are reliable.
The new axisymmetric formulations are used in a limited number of parametric studies, along with a more comprehensive number of plane strain parametric studies, to investigate the effect of important variables within a two-layer soil system. It was initially envisaged that an analytical model, capable of predicting the behaviour of a two-layer soil system, would be developed from the parametric study results. However, this proved to be difficult because of the complex nature of the mechanisms acting in the system and was abandoned in favour of a detailed numerical analysis of the system. In fact it is now thought that no simple analytical design method can fully describe the behaviour of a two-layer soil system.

Two fundamental difficulties with developing a simple analytical method are modelling the exact load distribution through the fill and predicting the precise deformed shape of the membrane. It is shown that the normal stresses at the fill base are not only influenced by the presence of reinforcement, but also by the shear strength of the underlying clay. For an increase in the clay shear strength the load spread angle (for which there is no unique definition) reduces considerably. Therefore, the load distribution characteristics through the fill layer are not even specific to the fill material properties and the use of a load spread parameter that is a function only of fill properties is erroneous. Previous design methods also make inappropriate assumptions concerning the fixity of the membrane in order to assess the deformed shape, but from the reported results herein such fixity is clearly spurious.

Much discussion has recently centred on the problem of making analytical predictions of the reinforcement mechanisms for small surface displacements. It has been shown that in a small strain, plane strain, analysis where geometric non-linear effects are explicitly excluded from the computation, the reinforcement causes only a nominal improvement in the load spread through the fill layer (hitherto often thought to be the most important mechanism), but, more importantly, it significantly reduces both the magnitude and the area of influence of the shear stresses acting on the clay surface. The reinforcement attenuates the large detrimental outward acting shear stresses that occur in the unreinforced case and increases the amount of beneficial inward acting shear beyond the vertically loaded area. It
is this shear stress mechanism of the reinforcement that is responsible for the majority of the loading improvement in the small strain analyses and, in fact, the large strain (plane strain and axisymmetric) analyses too.

Although the Houlbey et al. (1989) and Houlbey and Jewell (1990) limit equilibrium methods are rational procedures for estimating the structural improvement for small displacements, through the shear stress mechanism, they idealize the arrangement of stresses and consider the fill material beneath the footing to be in a state of active failure. It has been shown, however, that even at large footing displacements this is not necessarily the case, for both plane strain and axisymmetry. Moreover, there is undoubtedly an additional stiffness and strength within the two-layer system once large displacements develop and the geometric non-linear effects of the tensioned membrane and restraint mechanisms, identified as operating in the large strain analyses, are introduced. This highlights the need for any design method to consider the different reinforcement effects which exist at small and large displacements.

Despite the limitations of the Houlbey et al. (1989) and Houlbey and Jewell (1990) methods, the comparative analytical predictions of the ultimate footing load \( P_f \) are found to be reasonable approximations to the finite element results (at a footing displacement of \( 0.6B; R \)) for all of the small strain computations and, significantly, the plane strain and axisymmetric large strain reinforced computations. However, the predictions of \( P_f \) for the unreinforced large strain analyses and the values of the reinforcement force are less accurate, because of the afore-mentioned stress idealizations.

To obtain accurate analytical estimates of the reinforced \( P_f \), appropriate values of the load spread angle \( \beta \) must be selected. It is suggested that Table 12.1 may be used for the general case of a "normal" strength fill layer \( \phi_{ps} = 40^\circ \) of depth \( D_f \leq B; R \) overlying clay of strength \( s_u \), although these angles are approximations only \( (\pm 5^\circ) \). Since the parametric study does not cover all the possible combinations of \( s_u, \phi \), and \( D_f \), a definitive estimate of \( \beta \) cannot be made. However, the relationship between these properties and \( \beta \) is derived qualitatively from the Meyerhof (1974) theory, given in equation (7.6).
\[
\begin{array}{|c|c|c|}
\hline
S_u & \beta_{PS} & \beta_{As} \\
\hline
"Very soft" (\approx 10 kPa) & 60^\circ & 40^\circ \\
"Soft" (\approx 30 kPa) & 40^\circ & 30^\circ \\
"Firm" (\approx 70 kPa) & 30^\circ & 20^\circ \\
\hline
\end{array}
\]

Table 12.1: Approximate Plane Strain and Axisymmetric Load Spread Angles

A number of general conclusions relating to the behaviour of unreinforced and reinforced two-layer soil systems, deforming in plane strain and axisymmetry under the action of a single monotonic load, can be derived from the results of the large strain parametric study. It is clear that the presence of reinforcement has no appreciable influence on the load-displacement performance of the two-layer soil system within the elastic regime (i.e. for small surface displacements), compared to an unreinforced system. However, the reinforcement is shown to enhance the load bearing capacity of the system significantly, once plastic deformations occur, as observed frequently in practice and in the laboratory (e.g. Love (1984)).

The presence of reinforcement is found to have more effect for a thinner fill layer, than for a thicker layer. This conflicts with the implications of the Houlsby et al. (1989) and Houlsby and Jewell (1990) methods, which estimate greater lateral fill stresses for increases in fill thickness that consequently lead to a reduced unreinforced bearing capacity factor, and therefore the difference between the unreinforced and reinforced ultimate loads is larger for increasing fill thickness.

Using greater reinforcement stiffnesses results in improvements to the load-displacement response of the reinforced two-layer soil system, due to the increased contribution of the tensioned membrane mechanism. However, the benefit of increasing the reinforcement stiffness follows a pattern of diminishing returns, since larger shear stresses at the soil-reinforcement interface are generated, which induce higher reinforcement tensions, and consequently an arching effect develops in the fill. These additional shear stresses do not, in themselves, contribute to any structural improvement in the system. For
the cases studied a stiffness of the order $10^2 \text{kN/m}$ is perhaps the optimum range and there appears to be little further benefit beyond a modulus of around $3000 \text{kN/m}$. In fact the majority of the proprietary geosynthetics available, possess a stiffness within this suggested optimum range.

An important finding from the parametric study is that the length of the reinforcement needs to be sufficient to cover only the area of subgrade influenced by the normal and shear stresses acting at the interface. For a membrane of length just greater than this area of imposed load (i.e. $L = B + D_1 \tan \beta$), the shear stress mechanism and the tensioned membrane effect can both operate adequately and provide a completely reinforced system. This contradicts any requirements for the membrane to be firmly anchored outside of the load spread area in order to attain a completely reinforced system.

A slightly different approach to that used for the main parametric study is employed to assess the influence of the membrane-soil friction angle. A low friction angle geosynthetic ($\phi_i = 10^\circ$) is analysed, for a variety of different configurations, within a granular foundation. The results show that the displacements and soil deformations are not greatly increased, as might be expected, by introducing this potential plane of weakness. This illustrates the usefulness of placing impermeable low friction geosynthetics within foundations for fluid containment purposes.

The finite element formulations developed in this thesis are appropriate for the particular geotechnical problems considered. However, obvious improvements in the interface element formulation would be to incorporate a cohesive strength term and to allow large relative slip and even separation of the mating surfaces. These features should be included in any future work.
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