On the Thermodynamics of Porous Continua

by

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Abstract

The paper extends a thermomechanical approach to plasticity (hyperplasticity) to allow for constitutive modelling of porous continua. Strong emphasis is placed on the use of thermodynamic potentials for specification of the constitutive behaviour of a porous material. The entire behaviour can be derived from two scalar potential functions. The first is the energy potential, which consists of two components, for the skeleton and pore fluid, respectively. The second potential is the force potential, and is closely related to the dissipation function. This paper extends the applicability of the principles of hyperplasticity to problems involving large strains, fluid and heat flow in porous media, viscous effects, inertial and tortuosity effects.

1. Introduction

Collins and Houlsby (1997) and Houlsby and Puzrin (2000) developed “hyperplasticity” – an approach to constitutive modelling of plastic materials in which the entire response is determined by specification of two potential functions. Applications of this theory to in soil mechanics have been demonstrated in Puzrin \textit{et al.} (2001). There are many other areas of continuum mechanics where similar approaches have been made. For instance Ziegler (1977, 1983) develops theories for viscous materials. Many authors treat flow processes within a thermodynamic context, and frequently make use of a dissipation function. In Houlsby and Puzrin (2000) the special features of rate-independent materials have been the reason for a slightly different emphasis, when compared to most treatments of the subject.

This paper explores how the hyperplasticity approach can be generalised and set within the context of a wider variety of types of material behaviour. In particular, we place a strong emphasis on the use of two potential functions, and on the use of Legendre transformations to obtain alternative formulations for constitutive relationships.

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There are many approaches to the thermodynamics of materials. The approach adopted here is firmly rooted in the method of “Thermodynamics with Internal Variables” (T.I.V.) as classified by Maugin (1999).

1.1. Fluxes and Internal Variables

In the mechanics of dissipative materials using the internal variable approach, there are two classes of behaviour that cause dissipation. The first class involves fluxes, for instance flow in a porous medium or the flow of electrical current. In these cases the dissipation is associated with the spatial gradient of some variable (e.g. the hydraulic head for flow in a porous medium, the voltage for an electrical problem). The constitutive behaviour is usually (but not necessarily) described by a linear relationship between the flux and the spatial gradient.

The second class of dissipation is associated with the time variation of internal variables. The plasticity problems treated in Houlsby and Puzrin (2000) are of this character. Viscous behaviour is also described in this way.

Most texts, when treating the thermodynamics of dissipative continua, concentrate either on fluxes or on rates of change of internal variables. However, whilst the two problems have much in common, they also have important differences. Most obviously one involves a spatial variation and the other a time variation. It is tempting to treat both in the same way, and many (in fact most) texts adopt this approach, using for instance “generalised forces” and “generalised fluxes”. Here a slightly different approach will be adopted, keeping separate those variables associated with fluxes, and those associated with internal variables. In this way the different ways in which the two types of processes appear in the relevant equations can be made clearer.

1.2. Flow in porous media

Rather than considering the possibility of abstract, unspecified, fluxes, we find it more useful to consider a concrete example. The case that we consider is a very important problem in geomechanics and other fields, namely flow in a porous medium. This serves as a useful example because the flux itself has mass, which introduces a number of features to the problem that need careful treatment. The porous medium has to be treated as consisting of two phases, and there is a partition of the extensive quantities (e.g. internal energy, entropy) between the solid skeleton and fluid phases.

The subject of thermodynamics of porous continua has been treated extensively by Coussy (1995). The present formulation has a lot in common with his work. However, here we place a
much stronger emphasis on the use of the thermodynamic potentials for derivation of the entire constitutive behaviour of the porous material. As a result, we believe that the thermomechanical framework as formulated here contributes further to the understanding and modelling of porous continua. It allows a number of alternative formulations of the constitutive behaviour, depending on the kind of the energy potential (internal energy, Gibbs free energy, Helmholtz free energy or enthalpy) used for the solid skeleton and pore fluid. Unlike in the Coussy formulation, the same kind of potential can be used here for both the solid skeleton and pore fluid phases. This allows easy transitions between different formulations using standard Legendre-Fenchel transformations. Another important issue treated here is the fact that the dissipation function in more general cases is not, strictly speaking, a potential. Use of the force potential instead of the dissipation function contributes to the clarity and consistency of the formulation.

1.3. Large strains

Houlsby and Puzrin (2000) adopted a small strain formulation. The problem of coupled fluid and skeleton behaviour cannot be treated rigorously within the small strain framework. This is because there is a coupling between strains, fluid flow and density changes, whilst in the small strain formulation the density is treated as a constant. In the following therefore it is necessary to move to a large strain formulation. There is a choice between adopting a Lagrangian approach, in which the problem is formulated in terms of initial coordinates, and an Eulerian approach, in which it is formulated within the current coordinates. We adopt the Eulerian approach for much of the following development, since this allows a more direct interpretation of the variables. It will prove necessary, however, to make a transformation to Lagrangian variables for part of the analysis.

In the small strain approach, for convenience all the extensive quantities can defined per unit volume, as used by Houlsby and Puzrin (2000). Since the density was in effect constant this is exactly equivalent to use of extensive quantities per unit mass, but avoids a factor of the density appearing throughout the equations. In large strain analysis it is necessary to use extensive quantities per unit mass, as is more usual in thermodynamics, and we adopt this approach below.

2. Thermomechanical framework

As mentioned above, we adopt here an Eulerian approach to the description of a material undergoing large strain, i.e. the description of the material is based on the current co-ordinate
system. In this it will be necessary to distinguish between the time differential of a variable $x$ at a particular point in space, which we shall denote by $\partial x/\partial t = \dot{x}$, and the material derivative, which represents the rate of change for an element of the material, which has a current velocity $v_i$. We denote the material derivative by $dx/dt = \dot{x} = \dot{x} + x_j \dot{v}_i$. We can note from this definition that it follows that the chain rule applies to the convective derivative, e.g. the material derivative of $xy$ is $\dot{x}y + xy\dot{y}$.

2.1. Density definitions, velocities and balance laws

Consider a volume $V$ fixed in space bounded by a surface $S$. The unit outward normal to the boundary is $n_i$. The volume contains porous material with a skeleton material of density $\rho^s$ and with a porosity $n$ (volume of voids divided by total volume). Thus the mass of skeleton per unit total volume is $\rho = (1-n)\rho^s$. We should also note that $\rho$ is the “dry density” in soil mechanics terminology.

The velocity of the skeleton at any point is $v_i$, so that the mass flux of the skeleton per unit area is $\rho v_i$, and the outward mass flux per unit area from $V$ is $\rho v_i n_i$. For conservation of mass we can write that the rate of increase of mass within the volume, plus the outward mass flux is zero:

$$\int_V \dot{\rho} dV + \int_S \rho v_i n_i dS = 0$$

(1)

Applying the divergence theorem of Gauss, we can write this in local form:

$$\dot{\rho} + \left( \rho v_i \right)_j = \dot{\rho} + \rho_j v_i + \rho v_{i,j} = \ddot{\rho} + \rho v_{i,j} = 0$$

(2)

which establishes the link between the material rate of change of dry density and the dilatation rate.

We now allow for the possibility of fluxes of a pore fluid. We shall consider a pore fluid which has measure $w$ defined as mass of fluid per unit mass of skeleton material (i.e. the water content in soil mechanics terminology). The flux of the fluid mass is $m_i$ per unit area relative to the skeleton. The total flux vector of the fluid relative to the coordinate system is therefore $m_i + \rho w v_i$. The outward flux of the fluid across the boundary $S$ is therefore $(m_i + \rho w v_i) n_i$.

We note that the mass of fluid per unit volume of skeleton is $\rho w$. It follows that
\[ \rho w = np^w \]  

where \( \rho^w \) is the density of the fluid. The mass flux vector \( m_i \) can also be written as

\[ m_i = \rho^w w_i = \rho^w n (v_i^w - v_i) \]  

where \( w_i \) is the Darcy artificial seepage velocity and \( v_i^w \) is the actual absolute velocity of the fluid.

Noting that the mass of the fluid is conserved, there is a balance equation analogous to (1) of the form:

\[ \int_V \sum_i \left( \rho w \right) dV + \int_S (m_i + \rho w v_i) n_i dS = 0 \]  

which we can rewrite in local form by using the divergence theorem of Gauss to obtain the local conservation law:

\[ \dot{\rho} w + \rho \dot{w} + m_{i,j} + \rho w v_j + \rho w_j v_i + \rho w v_{i,j} = 0 \]  

or

\[ \rho \dot{w} + \rho w = m_{i,j} + \rho w v_{i,j} = 0 \]  

which by virtue of the skeleton mass conservation equation (2) becomes:

\[ \rho \dot{w} + m_{i,j} = 0 \]  

It is convenient to obtain a combined continuity equation for flow of the skeleton and pore fluid. First we can note \( \tilde{\rho} = \rho^s (1-n) - \rho^w n \) so that we can rewrite the mass continuity equation as:

\[ \tilde{\rho}^s (1-n) - \rho^s n + \rho^s (1-n) v_{i,j} = 0 \]  

We can also obtain by manipulation of (8):

\[ \tilde{\rho}^w n + \rho^w n - \left( \rho^w w_i \right)_j = \tilde{\rho} w = -wpv_{i,j} = -np^w v_{i,j} \]  

Finally dividing (9) by \( \rho^s \) and (10) by \( \rho^w \) and adding we obtain

\[ v_{i,j} + w_{i,j} + \frac{\rho^w}{\rho^w} n \tilde{\rho}^w + (1-n) \tilde{\rho}^s = 0 \]
If both the soil grains and the pore fluid are incompressible then this reduces to the simple form \( v_{i,i} + w_{i,i} = 0 \). Introducing \( v^w = 1/\rho^w \) and \( v^s = 1/\rho^s \) the continuity equation can also be written \( v_{i,i} + w_{i,i} = m_{i,j} v^w_{j,j} + \rho w v^w_{j,j} + \rho v^s_{j,j} \), where

\[
\frac{1}{\rho} = v = v^s + w v^w
\]  

(12)

2.2. Kinetic energy

As an extensive quantity, the kinetic energy of the whole matter enclosed in volume \( V \) may be written as the sum of the kinetic energies of the skeleton and of the fluid:

\[
K = \frac{1}{2} \int_V \rho (v_i)^2 dV + \frac{1}{2} \int_V \rho w (v^w_i)^2 dV
\]  

(13)

At this stage we are neglecting the effects of tortuosity, which accounts for the fact that the pore fluid must take a tortuous path between the skeleton particles, so that the average speed of the water particles is higher than the magnitude of the average velocity. We shall, however, show how the results can be modified later to take this into effect. Now consider the rate of change of kinetic energy in the volume \( V \), which can be written:

\[
\vec{K} = \int_V \frac{\partial}{\partial t} \left[ \frac{\rho v_i v_i}{2} + \frac{\rho w v^w_i v^w_i}{2} \right] dV + \int_S \left[ \frac{\rho v_i v_i}{2} \right] v_j n_j dS + \int_S \left[ \frac{\rho w v^w_i v^w_i}{2} \right] v^w_j n_j dS
\]  

(14)

The volume integral reflects changes of the kinetic energy with time in the volume, while the surface integrals account for the kinetic energy brought into the volume due to the skeleton and pore fluid movement through the surface. Applying the theorem of Gauss and grouping the resulting terms it is possible to obtain:

\[
\vec{K} = \int_V \rho v_i (\dot{v}_i + v_{i,j} v_j) dV + \int_V \left[ \rho + \rho v_{j,j} + \rho v_j v_j \right] \frac{v_i v_i}{2} dV + \int_V \rho w v^w_i (\dot{v}^w_i + v^w_{i,j} v^w_j) dV + \int_V \left[ \rho w + \rho \dot{v}^w + \rho w v^w_{j,j} + \rho w v^w_j v^w_j + \rho v^w_{j,j} w v^w_j \right] \frac{v^w_i v^w_i}{2} dV
\]  

(15)

Recalling the mass balance equations for skeleton and for fluid (2) and (6) respectively, we note that the second and fourth integrals vanish. We introduce also the definitions of the accelerations of the skeleton and fluid particles respectively:

\[
a_i = \ddot{v}_i = \dot{v}_i + v_{i,j} v_j
\]  

(16)
\[ a_i^w = \ddot{v}_i^w = \dot{v}_i^w + v_{i,j}^w v_j^w \]  \hspace{1cm} (17)

where the material derivative with respect to a fluid particle is denoted by \( \dot{x} + x_j v_i^w = \ddot{x} \). The expression for the rate of change of kinetic energy becomes:

\[ \tilde{K} = \int_V \left( \rho v_i a_i + \rho w v_i v_i^w \right) dV = \int_V \left( \rho a_i + \rho w v_i^w \right) v_i dV + \int_V m_i a_i^w dV \]  \hspace{1cm} (18)

2.3. Traction and body forces

The tractions (forces per unit area) on the skeleton on the fraction \((1-n)\) of the boundary \(S\) are \( t_i \), and the pressure in the pore fluid is \( p \), which acts on a fraction \( n \) of the boundary. The work done per unit area by the surroundings against the tractions on \( S \) is therefore \((1-n) t_i v_i \), and that done against the pore pressure is \(-n p n_i v_i^w \). There are also body forces arising from a gravitational field of strength \( g_i \). The work done per unit volume by the body forces on the skeleton is \( \rho v_i g_i \) and on the fluid is \( \rho w v_i^w g_i \). The heat flux per unit area is \( q_i \), so that the outward heat flux from \( S \) per unit area is \( q_i n_i \).

2.4. The First Law

The First Law of Thermodynamics states that there is a variable, called specific internal energy, such that the rate of increase of the internal energy in the volume plus the rate of change of the kinetic energy in this volume is equal to the sum of the rates of energy input at the boundaries (mechanical work and heat supply) plus the rate of work of the body forces in the volume\(^3\). We attribute a specific internal energy \( u^s \) to the skeleton and \( u^w \) to the pore fluid. The first law therefore becomes:

\(^3\) Note that, unlike many authors, we do not include a term for any body source of heat. In our view such a term is only necessary when insufficient variables have been included to provide a satisfactory description of the problem. For instance if a source term is included to represent energy released by a chemical reaction, it can be avoided by including variable representing the amounts of the various chemicals present, and their individual internal energies. On the other hand, the presence of such a source term is in effect a denial of the first law.
\[
\int \frac{\partial}{\partial t} \left( \rho u^s + \rho w u^w \right) dV + \int \left( \rho u^s v_i + \rho w u^w v_i^w \right) n_i dS + \\
\int \left( \rho a_i + \rho w a_i^w \right) v_i dV + \int m_i a_i^w dV = \\
\int \left( (1-n) \gamma v_j - n p n_j \right) n_i dS + \int \left( \rho v_i + \rho w v_i^w \right) g_i dV + \int (-q_i n_i) dS
\]  

(19)

where \( q_i \) is the heat flux. We can note that the tractions and pore pressure are related to the stresses by \( (1-n) \gamma v_j - n p n_j = \sigma_{ij} n_i \), so that \( (1-n) \gamma v_j = \sigma_{ij} v_j n_i + n v_j^w p n_j - p w_j n_j \) and we can rewrite the above as:

\[
\int \frac{\partial}{\partial t} \left( \rho u^s + \rho w u^w \right) dV + \int \left( \rho u^s v_i + \rho w u^w v_i^w \right) n_i dS + \\
\int \left( \rho a_i + \rho w a_i^w \right) v_i dV + \int m_i a_i^w dV = \\
\int \left( \sigma_{ij} v_j - p w_i - q_i \right) n_i dS + \int \left( \rho v_i + \rho w v_i^w \right) g_i dV
\]  

(20)

Applying the divergence theorem of Gauss we obtain a local form of the first law:

\[
\frac{\partial}{\partial t} \left( \rho u^s + \rho w u^w \right) + \left( \rho u^s v_i + \rho w u^w v_i^w \right) n_i = \\
\left( \sigma_{ij} v_j - p w_i - q_i \right) n_i + \left( \rho v_i + \rho w v_i^w \right) g_i - \\
\left( \rho a_i + \rho w a_i^w \right) v_i - m_i a_i^w
\]  

(21)

Expanding the differentials in the first row of equation (21) we obtain:

\[
\frac{d}{dt} \left( \rho u^s + \rho w u^w \right) + \left( \rho u^s v_{i,i} + \rho w u^w v_{i,i} \right) + \left( u^w m_i \right) i = \\
\frac{d}{dt} \left( \rho u^s + \rho w u^w \right) - \left( \rho u^s + \rho w u^w \right) + \left( u^w m_i \right) i = \\
\rho \tilde{u}^s + \rho \tilde{w}^w + u^w_i m_i
\]  

(22)

The second row of equation (21) may be transformed as follows:

\[
\left( \sigma_{ij} v_j - p w_i - q_i \right) n_i + \left( \rho v_i + \rho w v_i^w \right) g_i = \\
\left( \sigma_{ij} v_{j,i} + \sigma_{ij} v_{j,i} - p_j v_{j,i} - p w_{i,i} - q_{j,i} + \rho w v_i^w \right) g_i = \\
\left( \sigma_{ij} v_{j,i} + p (1 + w) g_j \right) h_j + \left( \rho w g_i - p_i \right) v_i - p w_{i,i} - q_{i,i}
\]  

(23)

We can then decompose \( v_{i,j} \) into its symmetric and antisymmetric parts, identifying the former as the strain rate and the latter as the vorticity tensor:
\[ d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \]  \hspace{1cm} (24)

\[ \omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i}) \]  \hspace{1cm} (25)

so that after substitution of (22)-(25) into (21) we can write:

\[ \rho \ddot{\mathbf{u}} + \rho \mathbf{w} \ddot{\mathbf{w}} + \mathbf{u} \dddot{\mathbf{u}} + \mathbf{m}_i = \]
\[ \left( \sigma_{ij,j} + \rho(1 + w)g_j - \rho a_j - \rho wa_j' \right) v_j + \sigma_{ij} \omega_{ji} + \]
\[ \sigma_{ij} d_{ij} + \left( \rho \left( g_i - a_i^w \right) - p_i \right) w_i - p w_{i,i} - q_i,i \]  \hspace{1cm} (26)

2.5. Equations of motion

No change in internal energy should, however, be caused by either a rigid body translation or rotation (due to the principle of material frame indifference), so that we can conclude that \( \left( \sigma_{ij,j} + \rho(1 + w)g_j - \rho a_j - \rho wa_j' \right) v_j = 0 \) and \( \sigma_{ij} \omega_{ji} = 0 \) for all \( v_j \) and \( \omega_{ji} \). These are of course the virtual work forms of the direct and rotational equilibrium conditions. From the latter it follows that the antisymmetric part of \( \sigma_{ij} \) must be zero, i.e. that \( \sigma_{ij} \) is symmetric. This condition is usually referred to as that of complementary shear stresses. From the former it follows that

\[ \sigma_{ij,j} + \rho(1 + w)g_j - \rho a_j - \rho wa_j' = 0 \]  \hspace{1cm} (27)

which can be recognised as the equations of motion (or the static equilibrium equations in the case of zero acceleration). Equation (27) expresses the momentum balance for the porous medium considered as a whole, and is derived as a part of formulation rather than postulated. However, this equation is not sufficient for description of the momentum balance of the pore fluid, which cannot be derived until some constitutive statement is made about interaction between the fluid and skeleton. The missing fluid balance equation will later also be derived as a part of the formulation.

In view of equation (27), equation (26) reduces to:

\[ \rho \ddot{\mathbf{u}} + \rho \mathbf{w} \ddot{\mathbf{w}} + \mathbf{u} \dddot{\mathbf{u}} + \mathbf{m}_i = \sigma_{ij} d_{ij} + \left( \rho \left( g_i - a_i^w \right) - p_i \right) w_i - p w_{i,i} - q_i,i \]  \hspace{1cm} (28)
2.6. The Second Law

The Second Law of Thermodynamics can be stated in a number of different ways. We state it here in the form that there exists a function of state, the specific entropy $s$, such that the rate of entropy production is non-negative. We attribute $s^s$ to the skeleton and $s^w$ to the pore fluid, so that the specific entropy of the whole medium is:

$$s = s^s + ws^w$$  \hspace{1cm} (29)

The flux of the entropy $\eta_i$ is defined by $\eta_i = q_i / \theta$, where $\theta$ is the temperature. Unlike the case for the flux of the pore fluid, the total amount of entropy is not conserved. This is expressed by writing the fundamental inequality for the entropy in the form:

$$\int \frac{\partial}{\partial t} \left( \rho s^s + \rho ws^w \right) dV + \int \left( \rho s^s v_j + \rho ws^w v_j^w \right) n_i dS \geq \int \left( -\frac{q_i}{\theta} n_i \right) dS$$  \hspace{1cm} (30)

The above equation states that the rate of increase of entropy within the volume, plus the convection of entropy across the boundary is greater than or equal to the entropy flux (from heat flow) into the volume. The additional entropy production is due to dissipative processes.

Applying the divergence theorem we obtain:

$$\frac{\partial}{\partial t} \left( \rho s^s + \rho ws^w \right) + \left( \rho s^s v_j + \rho ws^w v_j^w + \frac{q_i}{\theta} \right)_j \geq 0$$  \hspace{1cm} (31)

which can be written as

$$\frac{d}{dt} \left( \rho s^s + \rho ws^w \right) + \left( \rho s^s v_j + \rho ws^w v_j^w \right) + \left( \frac{q_i}{\theta} \right)_j = \hspace{1cm} (32)$$

$$\rho \tilde{s}^s + \rho w \tilde{s}^w + s_j m_j + \frac{q_{i,j}}{\theta} - \frac{q_{i,j}}{\theta^2} = \frac{\rho d^*}{\theta} \geq 0$$

Where $d^*$ is the specific dissipation, corresponding to irreversible part of entropy production, and must always be non-negative. We use here $d^*$ to distinguish this quantity from $d$ used in Houlsby and Puzrin (2000) for just the “mechanical dissipation”. The quantity $d^*$ includes also the term usually referred to as the “thermal dissipation”, which is due to the heat flux. The condition that the total dissipation be non-negative is slightly less restrictive than the requirement adopted by Houlsby and Puzrin (2000) that the mechanical dissipation be non-
negative. The statement of the Second Law, in the form presented by equation (32), can be traced back to the work of Green and Naghdi (1977), later developed by Rubin (1992).

2.7. Combining the first and second laws

We now combine equations (28) and (32) to eliminate the divergence of the heat flux and obtain:

\[
\begin{align*}
\rho \tilde{u}^s + \rho w \tilde{u}^w + \rho d^* &= \\
\sigma_{ij} d_{ij} - pw_{i,j} + \rho \theta \tilde{S}^s + \rho w \theta \tilde{S}^w - \eta_i \theta_j + \left( \rho^w (g_i - a_i^w) - p_j \right) v_i - \left( u_j^w - \theta s_j^w \right) n_i
\end{align*}
\]

which, by virtue of the continuity equation can be written as:

\[
\begin{align*}
\rho \tilde{u}^s + \rho w \tilde{u}^w + \rho d^* &= \\
\left( \sigma_{ij} + p \delta_{ij} \right) d_{ij} - \rho \theta \tilde{S}^s - \rho wp \tilde{S}^w + \rho \theta \tilde{S}^w - \eta_i \theta_j + \left( g_i - a_i^w - v^w p_j \right) m_i - \left( u_j^w - \theta s_j^w + p v_j^w \right) n_i
\end{align*}
\]

Defining the total internal energy per mass of skeleton as \( u = u^s + wu^w \) we then obtain:

\[
\tilde{u} + d^* = \frac{1}{\rho} \left( \sigma_{ij} + p \delta_{ij} \right) d_{ij} - \rho \tilde{S}^s - wp \tilde{S}^w + \theta \tilde{S}^w + u^w \tilde{w} + \frac{1}{\rho} \left( g_i - a_i^w - v^w p_j \right) m_i - \frac{1}{\rho} \left( u_j^w - \theta s_j^w + p v_j^w \right) n_i - \eta_i \theta_j
\]

The left hand is clearly the sum of a stored term (\( \tilde{u} \)) and a dissipated term (\( d^* \)). It is tempting therefore to identify it with the total energy input, but this would be incorrect, as that is represented by \( \tilde{u} \) itself. Nevertheless, in the following it will be shown that writing of the energy equation as a sum of the stored and dissipated terms in the left hand side leads to a very convenient framework for introduction of the energy potentials.

The right hand side of (35) includes three types of term. The first involves the strain rate. The second type involves material differentials, and the third involve fluxes. The presence of the strain rate poses a problem within the Eulerian formulation, as it does not prove to be possible to express the strain rate as a material derivative of any observable quantity. This problem can be avoided by adopting a Lagrangian formulation. We can rewrite:

\[
\frac{1}{\rho} \left( \sigma_{ij} + p \delta_{ij} \right) d_{ij} = \frac{1}{\rho_o} \left( \pi_{ij} + p^e \delta_{ij} \right) \Delta_{ij}
\]
where \( \pi_{ij} \) is the Piola-Kirchhoff stress tensor and \( \Delta_{ij} \) is the Green-Lagrange strain defined by

\[
2\Delta_{ij} = P_{ki} P_{kj} - \delta_{ij}
\]

where \( P_{ij} = \partial x_i / \partial X_j \) and \( x_i \) and \( X_i \) are the current (Eulerian) and initial (Lagrangian) coordinates of a material point measured in a Cartesian system. The initial dry density is \( \rho_o \). It can be shown that \( \pi_{ij} = \det(P_{ij}) P^{-1}_{ik} \sigma_{kl} P^{-1}_{jl} \) and \( \Delta_{ij} = P_{ki} d_{kl} P_{lj} \). It is also necessary to introduce the variable \( p^t = p \det(P_{ij}) P^{-1}_{ik} P^{-1}_{jk} \) which is the transformation of the pore pressure to the Lagrangian coordinate system. We note that in the Lagrangian coordinates no distinction is necessary between the time and material derivatives so that

\[
\dot{\Delta}_{ij} = \partial \Delta_{ij} / \partial t = d \Delta_{ij} / dt = \ddot{\Delta}_{ij}
\]

In principle it would be possible to transform all the other variables to Lagrangian coordinates too, but this has the disadvantage that physical meaning is lost. Since these transformations are not strictly necessary for the following argument we shall leave the remaining terms in their Eulerian form.

We can now write (35) as:

\[
\tilde{u} + d^* = \frac{1}{\rho_o} \left( \pi_{ij} + p^t \delta_{ij} \right) \Delta_{ij} - p \tilde{v}^s - w p \tilde{v}^w + \theta \tilde{v}^w + w \theta \tilde{s}^w + u^w \tilde{w} + \\
\frac{1}{\rho_o} \left( g_{ij} - a_i^w - v^w p_{ij} \right) m_i - \frac{1}{\rho_o} \left( u_i^w - \theta s_i^w + p v_i^w \right) m_i - \frac{1}{\rho_o} \eta_i \theta_j
\]

(37)

2.8. The internal energy function

Following usual practice that the internal energy is assumed to be a function of the entropy and specific volume, we now adopt the hypothesis that the internal energy is a function of the strains, the water content, the extensive quantities \( v^s, s^s, v^w \) and \( s^w \), and certain internal variables \( \alpha_{ij} \). We assume that the function \( \alpha \) can be decomposed in the form:

\[
u = u \left( \Delta_{ij}, \alpha_{ij}, v^s, s^s, w, v^w, s^w \right) \]

\[
u = u \left( \Delta_{ij}, \alpha_{ij}, v^s, s^s \right) + w u^w \left( v^w, s^w \right)
\]

(38)

so that:

\[
\tilde{u} = \frac{\partial u}{\partial \Delta_{ij}} \Delta_{ij} + \frac{\partial u}{\partial \alpha_{ij}} \alpha_{ij} + \frac{\partial u}{\partial v^s} \tilde{v}^s + \frac{\partial u}{\partial s^s} \tilde{s}^s + \frac{\partial u}{\partial w} \tilde{w} + \frac{\partial u}{\partial v^w} \tilde{v}^w + \frac{\partial u}{\partial s^w} \tilde{s}^w
\]

\[
= \frac{\partial u}{\partial \Delta_{ij}} \Delta_{ij} + \frac{\partial u}{\partial \alpha_{ij}} \alpha_{ij} + \frac{\partial u}{\partial v^s} \tilde{v}^s + \frac{\partial u}{\partial s^s} \tilde{s}^s + u^w \tilde{w} + \frac{\partial u}{\partial v^w} \tilde{v}^w + \frac{\partial u}{\partial s^w} \tilde{s}^w
\]

(39)
2.9. The dissipation function and force potential

We also postulate that the dissipation is a function of the same state variables, but also of $\alpha_{ij}$ (rate of change of internal variable) and of the fluxes, i.e.:

$$d^* = d^*\left(\Delta_{ij}, \alpha_{ij}, v^s, s^s, w, v^w, s^w, \tilde{\alpha}_{ij}, m_i, \eta_i\right)$$  \hspace{1cm} (40)

We next introduce the important assumption that $d^*$ is a pseudo-potential for “generalised forces”. This assumption is equivalent to the following expression:

$$d^* = \frac{\partial z}{\partial \alpha_{ij}}\alpha_{ij} + \frac{\partial z}{\partial m_i}m_i + \frac{\partial z}{\partial \eta_i}\eta_i$$  \hspace{1cm} (41)

where $z$, which we shall call the force potential, is a function of the same variables as $d^*$:

$$z = z\left(\Delta_{ij}, \alpha_{ij}, v^s, s^s, w, v^w, s^w, \tilde{\alpha}_{ij}, m_i, \eta_i\right)$$  \hspace{1cm} (42)

In fact this proves not to be a particularly onerous condition, since most plausible expressions for the dissipation are in fact pseudo-potentials. Maugin (1999) makes use of the equivalent of $z$, which he calls a “dissipation potential”.

2.10. Constitutive equations

Substituting equations (39) and (41), into (35), and collecting terms we obtain:

$$0 = \left(\frac{1}{\rho_o} \left(\pi_{ij} + pL \delta_{ij}\right) - \frac{\partial u^s}{\partial \alpha_{ij}}\right)\tilde{\Delta}_{ij} + \left(- p - \frac{\partial u^s}{\partial v^s}\right)\tilde{v}^s + \left(\theta - \frac{\partial u^s}{\partial s^s}\right)\tilde{s}^s$$

$$+ \frac{w}{\rho} \left(- p - \frac{\partial u^w}{\partial v^w}\right)\tilde{v}^w + \frac{w}{\rho} \left(\theta - \frac{\partial u^w}{\partial s^w}\right)\tilde{s}^w$$

$$+ \frac{\partial u^s}{\partial \alpha_{ij}} - \frac{\partial z}{\partial \alpha_{ij}}\tilde{\alpha}_{ij}$$

$$+ \frac{1}{\rho} \left(g_{ij} - a_{ij} + v^w p_{j}\right)\frac{\partial z}{\partial m_i}m_i + \left(- \frac{1}{\rho} \theta_{j} - \frac{\partial z}{\partial \eta_i}\right)\eta_i$$

$$- \frac{1}{\rho} \left(u_{i,j}^w - \theta s_{i,j}^w + p v_{i,j}^w\right)\eta_i$$  \hspace{1cm} (43)

Now equation (43) should be satisfied for any admissible combination of $\tilde{\Delta}_{ij}, \tilde{\alpha}_{ij}, \tilde{v}^s, \tilde{s}^s, \tilde{v}^w, \tilde{s}^w, \eta_i$ and $m_i$, and since all these quantities are independent of each other, each term
in equation (43) has to be equal to zero independently. Because the internal energy function (38) is independent of \( \Delta_{ij}, \bar{v}^s, \bar{s}^s, \bar{v}^w \) and \( \bar{s}^w \), from the first two rows of (43) it follows that:

\[
\frac{1}{\rho_o} \left( \pi_{ij} + p \delta_{ij} \right) = \frac{\partial u^s}{\partial \Delta_{ij}} \tag{44}
\]

\[-p = \frac{\partial u^s}{\partial v^s} \tag{45}\]

\[\theta = \frac{\partial u^s}{\partial s^s} \tag{46}\]

\[-p = \frac{\partial u^w}{\partial v^w} \tag{47}\]

\[\theta = \frac{\partial u^w}{\partial s^w} \tag{48}\]

Considering now a spatial gradient of \( u^w \), we can obtain:

\[u^w_j = \frac{\partial u^w}{\partial v^w} v^w_j + \frac{\partial u^w}{\partial s^w} s^w_j = -p v^w_j + \theta s^w_j \tag{49}\]

so that the fifth row of (43) is identically zero.

Equations (44) to (48) express an essential property of the internal energy function: that it is a potential for stresses and temperature. The basic form of these relationships is well known from hyperelasticity, but the particular expressions here deserve some comment. First note that the intensive quantities of pore pressure and temperature each appear as a partial derivative of the internal energy of the skeleton and of the pore fluid. The fact that both derivatives are related the same value of the intensive variable reflects in effect an assumption of an intimate mixing of the two phases. The temperature of the solids and fluid is assumed to be the same, and the pore pressure acts equally on the solids and the fluid.

Equation (44) embodies Terzaghi’s principle of effective stress for a porous medium. It demonstrates that (for the choice of kinematic variables we have made) the quantity that is work conjugate to the strain rate is not the total stress \( \sigma_{ij} \) but the effective stress \( \bar{\sigma}_{ij} = \sigma_{ij} + p \delta_{ij} \) (the positive sign appears because we have followed the tensile positive
convention usual in continuum mechanics for the stresses, whilst the pore pressure is positive in compression). The corresponding definition of the Lagrangian effective stress is given by the following equation:

\[ \bar{\pi}_{ij} = \pi_{ij} + p^T \delta_{ij} \]  

(50)

Unfortunately, an argument similar to that used to develop equations (44) to (48) above cannot be applied to the terms in third and fourth rows of equation (43), because the \( z \) function does depend on \( \tilde{c}_{ij}, \eta_i \) and \( m_i \). Assuming independence of the fluxes and of the rate of the internal variable, only the weaker conditions \( \left( \frac{\partial u^x}{\partial \alpha_{ij}} + \frac{\partial z}{\partial \alpha_{ij}} \right) \tilde{\alpha}_{ij} = 0 \):

\[ \left( \theta_{ij} + \rho \frac{\partial z}{\partial \eta_i} \right) \eta_i = 0 \quad \text{and} \quad \left( v^w p_{ij} - g_i + a_i^w \right) + \rho \frac{\partial z}{\partial m_i} m_i = 0 \]  

can be formally derived from equation (43).

However, at this stage we restrict ourselves to analysis of models to which stricter conditions (than those described above) can be applied:

\[ \frac{\partial u}{\partial \alpha_{ij}} + \frac{\partial z}{\partial \tilde{\alpha}_{ij}} = 0 \]  

(51)

\[ \theta_{ij} = -\rho \frac{\partial z}{\partial \eta_i} \]  

(52)

\[ v^w p_{ij} - g_i + a_i^w = -\rho \frac{\partial z}{\partial m_i} \]  

(53)

Equation (51) is Ziegler’s orthogonality condition, defining visco-plastic constitutive behaviour, equation (52) is (for an appropriate form of \( z \)) the Fourier heat conduction law, while equation (53), is in fact the missing equation of motion of the pore fluid, which becomes more obvious after it is rewritten as follows:

\[ \left( -p \delta_{ij} \right)_{ij} + \rho^w g_j - \rho^w a_j^w - \rho^w \rho \frac{\partial z}{\partial m_j} = 0 \]  

(54)

where the third term in the left part can be identified as a “drag” force.

When inertial effects can be neglected, and with an appropriate choice of force potential \( z \), equation (53) becomes Darcy’s law for fluid flow (for a fluid of constant density \( \rho^w \)).
3. Discussion

Equations (44)-(48) and (51)-(53) represent a complete set of the constitutive relationships describing the material, which is therefore defined entirely by specification of two scalar potential functions \( u \) and \( z \) in equations (38) and (42), respectively. Equations (51)-(53) are sufficient, but not necessary, to ensure that the laws of thermodynamics are obeyed.

Note that the entire formulation of the constitutive behaviour is based on the following principles:

- the mass conservation laws (1) and (5);
- the First Law of Thermodynamics (19);
- the Second Law of Thermodynamics (30);

and the following assumptions:

- the existence of the internal energy function (38) independent of rigid body translation and rotation;
- the existence of the quasi-homogeneous dissipation function (40) satisfying equation (41);
- the two above functions should be related through Ziegler’s orthogonality condition (51) and the Onsager reciprocity relationships for the fluxes which follow directly from (52) and (53).

Needless to say, the above formulation of the constitutive behaviour is guaranteed to satisfy the laws of thermodynamics. We make no secret of the fact, however, that we have introduced some additional, more restrictive assumptions. It is for the reader to decide whether these restrictions reduce the scope for constitutive modelling to such an extent that the materials that can be described are no longer realistic. We address below, however, some of the advantages that follow from adopting the slightly more restrictive approach. It is our belief that in fact a very wide variety of material response can be described within this framework. Furthermore we are not aware of any specific counter examples from the physical world, which provide clear evidence that the restrictions imposed above are invalid.

4. The complete formulation

We summarise the position we have arrived at as follows. The first step is to specify a constitutive model through internal energy and force potential functions:
\[
\begin{align*}
u &= u_{ij} \alpha_{ij} \nu_s \nu_w \psi_s \psi_w \\
&= u_s \alpha_{ij} \nu_s \nu_w + wu \psi_w \psi_s \\
z &= z(\Delta_{ij} \alpha_{ij} \nu_s \nu_w \psi_s \psi_w \tilde{\alpha}_{ij} m_i \eta_i) 
\end{align*}
\] (55) (56)

Using these functions, the differential relationships in Table 1 are applied. The variables involved in the solution (with the numbers of variables for vector and tensor quantities) are \(x_i\) (3), \(v_i\) (3), \(v_i^w\) (3), \(a_i\) (3), \(a_i^w\) (3), \(p_{ij}\) (9), \(\Delta_{ij}\) (6), \(\pi_{ij}\) (6), \(\alpha_{ij}\) (6), \(\rho, \psi, \psi_s, \psi_w, \psi_w, s^w, p, p^x, \theta, m_i\) (3) and \(\eta_i\) (3), that is 57 variables in all. Specification of initial and boundary conditions completes the formulation.

4.1. Modifications to account for tortuosity

As mentioned above, the effects of tortuosity have been neglected in the above derivations. It is usual to account for such effects by introducing a tortuosity factor \(a\) (see e.g. Coussy, 1995), which is the ratio between average of the squared microscopic relative velocity of the fluid with respect to the skeleton, and the square of the average of the same quantity. It is straightforward to show that \(a\) is a factor always greater than or equal to unity. When this factor is included, the expression for the kinetic energy becomes:

\[
K = \frac{1}{2} \int_{\Omega} \rho(v_r)^2 dV + \frac{1}{2} \int_{\Omega} \rho_{w}(v_{r}^w)^2 + (a-1)(v_{r}^w)^2 dV 
\] (57)

where \(v_r^w = v_i^w - v_i\) is the macroscopic relative velocity of the fluid with respect to the skeleton.

When the additional terms due to this change are followed through to equation (37), the only change necessary is to replace \(a_i^w\) in the first term on the second line with an effective acceleration term \(a_i^{we}\), which is defined as:

\[
a_i^{we} = a_i^w + (a-1)v_{r}^w + \frac{1}{2} \bar{a} v_{r}^w
\] (58)

Note that the term in \(a_i^w\) in the equation of motion (27) is not altered. In Table 1 the \(a_i^w\) term in the first equation of motion is unaltered, but that in the second equation of motion is modified to \(a_i^{we}\). Because there is now the additional variable \(a\) in the problem, a further
equation is now required. This must be a constitutive relationship for the tortuosity factor $a$, which could for instance be expressed as a function of the porosity. Berryman (1980) suggests the expression \( a = (n + 1)/2n \) for a matrix of spherical particles. In terms of the variables in Table 1 this would become \( a = (\rho \nu v^w + 1)/2\rho \nu v^w \). Note that this is the only constitutive equation which does not arise directly from the specification of the energy and dissipation functions.

5. Legendre-Fenchel transforms

In classical thermodynamics, in addition to the specific internal energy \((u)\), three other energy functions are defined: specific Helmholtz free energy \((f)\), specific enthalpy \((h)\) and specific Gibbs free energy \((g)\). These functions are related to the specific internal energy \((55)\) through a series of Legendre-Fenchel Transformations (for details on Legendre transformations in this context see for example the appendix to Collins and Houlsby (1997)), as shown in Table 2. As demonstrated in Table 2, all the derived energy functions \(e\) can be also decomposed into the specific energy functions \(e^s\) and \(e^w\) attributed to the skeleton and fluid, respectively. Moreover, by the virtue of equations (12) and (29), these parts can be transformed independently. The constitutive equations in the last row of Table 2 follow directly as properties of the corresponding Legendre-Fenchel Transformations.

The choice of which formulation to use will depend on the application in hand. For instance the four forms of the energy potential in classical thermodynamics are adopted in different cases (e.g. isothermal problems, adiabatic problems etc.).

The dissipation function formulation can be also adjusted to the chosen energy formulation, simply by expressing the force potential \(z\) through corresponding variables:

\[
\begin{align*}
z &= z^u \left( \Lambda, \alpha, v, s, w, v^w, s^w, \tilde{\alpha}, m, \eta \right) \\
z &= z^f \left( \Lambda, \alpha, v^s, \tilde{\theta}, w, v^w, \tilde{\alpha}, m, \eta \right) \\
z &= z^h \left( \tilde{\pi}, \alpha, p, s^s, w, s^w, \tilde{\alpha}, m, \eta \right) \\
z &= z^g \left( \tilde{\pi}, \alpha, p, \tilde{\theta}, w, \tilde{\alpha}, m, \eta \right)
\end{align*}
\]

(59)

In principle it would be possible to define the dissipation function in terms of a different set of variables than the energy function, but only in rather particular circumstances might this be useful.

A Legendre-Fenchel transformation from the force potential to a “flow potential” \(w\) is also possible (see Houlsby and Puzrin, 2001), but we do not pursue this here.
6. Small strain formulation

The displacement vector of a point with initial coordinates $X_i$ is defined by $u_i = x_i - X_i$, so that the deformation gradient can be expressed as $P_{ij} = \delta_{ij} + u_{i,j}$. The assumption of small strains is equivalent to $\|u_{i,j}\| \ll 1$, therefore $\det(P_{ij}) \approx 1 + u_{i,i}$ and, after the higher order terms are neglected, the following simplifications can be applied to the formulation in Table 1:

- $2\Delta_{ij} = P_{ki}P_{kj} - \delta_{ij} \approx u_{i,j} + u_{j,i} = 2\varepsilon_{ij}$, where $\varepsilon_{ij}$ is the linearized strain tensor;
- $\pi_{ij} = \det(P_{ij})P_{ik}^{-1}\sigma_{kl}P_{jl}^{-1} \approx \sigma_{ij}$;
- $p^f = p \det(P_{ij})P_{ik}^{-1}P_{kj}^{-1} \approx p$.

The linearized strain tensor can in some circumstances be decomposed into elastic and plastic components: $\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p$, and the kinematic internal variable $\alpha_{ij}$ can be associated with the plastic strain tensor $\varepsilon_{ij}^p$.

7. Example

The following example describes a conventional thermo-poro-elasto-plastic model, which can be applied to describe the small strain behaviour of a saturated isotropic frictional granular material. The constitutive behaviour is completely defined by the following two potential functions. The first one is Gibbs free energy function $g = g^s + wg^w$, which we assume to have constant, linear and quadratic terms, written in the following form so that certain constants retain their usual meaning:

\[
\begin{align*}
g^s &= g^s_0 + (p - p_0)v^s_0 - (\theta - \theta_0)s^s_0 \\
&\quad - \frac{(p - p_0)^2}{2K^s}v^s_0 + 3\alpha^s(\theta - \theta_0)(p - p_0)v^s_0 - c^s_p(\theta - \theta_0)^2 \quad (60) \\
&\quad - \frac{1}{\rho_0} \left( \frac{1}{3K} \sigma_{ij}^e \sigma_{ij}^e + \frac{1}{2G} \sigma_{ij}^e \sigma_{ij}^e + \alpha(\theta - \theta_0)\sigma_{kk}^e + \sigma_{ij}^e \alpha_{ij} \right)
\end{align*}
\]

\[
\begin{align*}
g^w &= g^w_0 + (p - p_0)v^w_0 - (\theta - \theta_0)s^w_0 \\
&\quad - \frac{(p - p_0)^2}{2K^w}v^w_0 + 3\alpha^w(\theta - \theta_0)(p - p_0)v^w_0 - c^w_p(\theta - \theta_0)^2 \quad (61)
\end{align*}
\]

where initial values are denoted with the subscript “0”, and the constants have the following physical meanings:
$K^s$ and $K^w$ are the isothermal bulk moduli of the skeleton particles and fluid, respectively;

$3\alpha^s$ and $3\alpha^w$ are the volumetric thermal expansion coefficients of the skeleton particles and fluid, respectively;

$c_p^s$ and $c_p^w$ are the mass heat capacity at constant pressure $p_o$ of the skeleton particles and fluid, respectively;

$K$ and $G$ are the isothermal bulk and shear moduli of the skeleton matrix, respectively;

$\alpha$ is the linear thermal expansion coefficient of the skeleton matrix.

The second function required is the force potential:

$$z^g = \frac{\mu \sigma_{ii}}{\rho_0} \sqrt{\tilde{\alpha}_{ij} \tilde{\alpha}_{ij}} + \frac{\Lambda}{\rho_0} \left( 3\beta \sqrt{\tilde{\alpha}_{ij} \tilde{\alpha}_{ij}} + \tilde{\alpha}_{ii} \right) + \frac{\nu^w}{2p_k m_i m_j} + \frac{\theta}{2p_k \eta_i \eta_j},$$

where $\Lambda$ is a Lagrangian multiplier associated with a dilation constraint. Houlsby and Puzrin (2000) discuss the use of kinematic constraints in this context. The constants have the following physical meanings:

$\kappa_m$ is the permeability coefficient;

$\kappa_\lambda$ is the thermal conductivity coefficient;

$\mu$ and $\beta$ are the coefficients related to the effective angles of friction $\phi'$ and dilation $\nu$

obtained in triaxial compression:

$$\beta = \frac{2\sqrt{2} \sin \psi}{\sqrt{3(3 - \sin \psi)}}$$

$$\mu + \beta = \frac{2\sqrt{2} \sin \phi'}{\sqrt{3(3 - \sin \phi')}}$$

The following constitutive relationships can be derived from the above formulation using equation from the last column of Table 2:

$$\varepsilon_{ii} = \frac{\tilde{\sigma}_{ii}}{3K} + 3\alpha(\theta - \theta_0) + \alpha_{ii}; \varepsilon_{ij}' = \frac{\tilde{\sigma}_{ij}'}{2G} + \alpha_{ij}'$$

$$\frac{\nu^s - \nu_0^s}{\nu_0^s} = 3\alpha^s(\theta - \theta_0) - \frac{\nu - \nu_0}{K^s}$$
\[
\frac{v^w - v_0^w}{v_0^w} = 3\alpha^w (\theta - \theta_0) - \frac{p - p_0}{K^w} \\
\]

\[
s^s = s_0^s + c_p^s \frac{(\theta - \theta_0)}{\theta_0} - 3\alpha^s (p - p_0)h_0^s + \frac{\alpha\sigma_{kk}}{\rho_0} \\
\]

\[
s^w = s_0^w + c_p^w \frac{(\theta - \theta_0)}{\theta_0} - 3\alpha^w (p - p_0)h_0^w \\
\]

Equations (65) represent decomposition of the strain tensor into elastic and plastic components, where the elastic part is defined by conventional thermoelasticity. Equations (66) and (68) give thermoelastic relationships for skeleton, while equations (67) and (69) represent classical thermoelastic relationships for fluid.

Evolution equations for plastic strains are obtained by defining generalized stress

\[ \bar{\chi}_{ij} = -\rho_o \frac{\partial g^s}{\partial \alpha_{ij}^s} = \bar{\sigma}_{ij} \] and the dissipative generalized stress \[ \chi_{ij} = \rho_o \frac{\partial \bar{z}^R}{\partial \alpha_{ij}} \], so that

\[
\chi_{ij} = (\mu\bar{\sigma}_{ii} + 3\Lambda\beta) \frac{\bar{\alpha}_{ij}'}{\sqrt{\bar{\alpha}_{ij}'\bar{\alpha}_{ij}'}} ; \chi_{ii} = 3\Lambda \\
\]

and by eliminating \( \bar{\chi}_{ij}' \) and \( \Lambda \) from equations (70) we obtain the equation of the yield surface in the dissipative generalized stress space:

\[
y^S(\bar{\sigma}_{ij}, \chi_{ij}) = \sqrt{\chi_{ij}'\chi_{ij}'} - (\mu\bar{\sigma}_{ii} + \beta\chi_{ii}) = 0 \\
\]

The flow rule \( \bar{\alpha}_{ij} = \bar{\lambda} \frac{\partial y^R}{\chi_{ij}} \) (where \( \bar{\lambda} \) is a Lagrangian multiplier) follows from the properties of the Legendre-Fenchel Transformation \( y^S = \chi_{ij} \bar{\alpha}_{ij} / \rho_0 - z^S = 0 \) relating the yield surface to the force potential. When Ziegler’s orthogonality condition \( \chi_{ij} = \bar{\lambda}_{ij} \) is applied to the equations of the yield surface and flow rule, it becomes clear that the behaviour described is equivalent to that of a perfectly plastic model with the Drucker-Prager failure cone defined by an effective angle of internal friction \( \phi' \), and a non-associated flow rule with the plastic potential cone defined by an angle of dilation \( \psi \).

Finally, from equations (52) and (53), after neglecting inertial effects and substituting \( \eta_i = q_i / \theta \) and \( m_i = w_i / v^w \), it follows that:
\[ q_i = -k_i \theta_i \quad (72) \]

\[ w_i = -\dot{k}_m (v \cdot p_i - g_i) \quad (73) \]

Equation (72) is the isotropic Fourier heat conduction law, while for constant \( v \cdot p \) equation (73) becomes the isotropic version of Darcy’s law for fluid conduction.

It is remarkable that a model which includes elasticity, plasticity (including dilatation), thermal effects, pore fluid flow and heat conduction can be entirely specified through just two potentials. It is also worth observing that the role of each term in the potentials, and how it relates to the relevant physical phenomenon, is fairly transparent.

8. Conclusions

The theoretical framework presented in this paper extends applicability of the principles of hyperplasticity to the problems involving:

- large strains;
- fluid flow in the porous media;
- heat flow in the porous media;
- viscous effects;
- inertial and tortuosity effects.

Apart from this generalization, the proposed framework places slightly less stringent restrictions on the class of constitutive models in terms of the requirements of the Second Law of Thermodynamics than had been used in our earlier work. Within the framework described by Houlsby and Puzrin (2000), the fact that the mechanical dissipation was required to be non-negative resulted in a condition which is more stringent than the Second Law. Within the present framework it is the total dissipation (including dissipation due to heat and fluid fluxes) that has to be non-negative, which is equivalent to the Second Law.

As in the standard hyperplastic approach, the entire constitutive behaviour is completely defined by specification of two scalar potential functions. However, in the generalized framework these functions also include the properties related to the different phases of the media and their interaction. The fluid and heat conduction laws are also built into these potentials, completing description of the constitutive behaviour of the complex media.
9. References


**Table 1: Summary of equations**

<table>
<thead>
<tr>
<th>Equation type</th>
<th>Equation</th>
<th>Number of equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Differential of free energy (skeleton)</td>
<td>( \pi_{ij} + p^L \delta_{ij} = \rho_o \frac{\partial u^s}{\partial \Delta_{ij}} )</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>(- p = \frac{\partial u^s}{\partial v^s} )</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>( \theta = \frac{\partial u^s}{\partial s^s} )</td>
<td>1</td>
</tr>
<tr>
<td>Differentials of free energy (pore fluid)</td>
<td>(- p = \frac{\partial u^w}{\partial v^w} )</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>( \theta = \frac{\partial u^w}{\partial s^s} )</td>
<td>1</td>
</tr>
<tr>
<td>Ziegler’s orthogonality condition</td>
<td>( \frac{\partial u^s}{\partial \alpha_{ij}} + \frac{\partial \alpha_{ij}}{\partial \alpha_{ij}} = 0 )</td>
<td>6</td>
</tr>
<tr>
<td>Heat conduction law</td>
<td>( \theta = \frac{-\rho}{\partial z/\partial \eta_l} )</td>
<td>3</td>
</tr>
<tr>
<td>Mass balance equations</td>
<td>( \tilde{\rho} + \rho v_{i,j} = 0 )</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>( \rho \tilde{\omega} + m_{i,j} = 0 )</td>
<td>1</td>
</tr>
<tr>
<td>Second Law of Thermodynamics</td>
<td>( \theta \left( \frac{\tilde{\omega}}{\rho} + \frac{\tilde{z}}{\rho} \right) - \frac{1}{\rho} \left( \frac{\tilde{w}}{\rho} \right) m_i + \frac{1}{\rho} \eta_l = 0 )</td>
<td>1</td>
</tr>
<tr>
<td>Equations of motion</td>
<td>( \left( P_{ik} \pi_{ik} \right)<em>{i,j} + \rho_o (1 + w) g</em>{i,j} = \rho_o \left( a_j + wa^w \right) )</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>( \nu^w p_{i,j} - g_i + a_i^w = \frac{-\rho}{\partial z/\partial m_i} )</td>
<td>3</td>
</tr>
<tr>
<td>Strain definition</td>
<td>( 2\Delta_{ij} = P_{ik} P_{kj} - \delta_{ij} )</td>
<td>6</td>
</tr>
<tr>
<td>Deformation gradient</td>
<td>( P_{ij} = \frac{\partial X_i}{\partial X_j} )</td>
<td>9</td>
</tr>
<tr>
<td>Skeleton velocity</td>
<td>( v_i = \tilde{x}_i )</td>
<td>3</td>
</tr>
<tr>
<td>Skeleton acceleration</td>
<td>( a_i = \tilde{v}_i )</td>
<td>3</td>
</tr>
<tr>
<td>Fluid acceleration</td>
<td>( a_i^w = \tilde{v}<em>i^w + v</em>{i,j} v_j^w )</td>
<td>3</td>
</tr>
<tr>
<td>Mass flux definition</td>
<td>( m_i = \rho w \left( v_i^w - v_i \right) )</td>
<td>3</td>
</tr>
<tr>
<td>Density definition</td>
<td>( 1/\rho = v^s + w v^w )</td>
<td>1</td>
</tr>
<tr>
<td>Lagrangian pore pressure</td>
<td>( p^L = \rho \det(P_{ik}) P_{ik}^{-1} P_{ik}^{-1} )</td>
<td>1</td>
</tr>
<tr>
<td><strong>Total number of equations</strong></td>
<td></td>
<td>57</td>
</tr>
</tbody>
</table>
Table 2: Energy potentials for use in large strain continuum mechanics

<table>
<thead>
<tr>
<th>Internal energy</th>
<th>Helmholtz free energy</th>
<th>Enthalpy</th>
<th>Gibbs free energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u = u(\Delta_{ij}, \alpha_{ij}, v^s, s^w, w, v^w, s^w) )</td>
<td>( f = f(\Delta_{ij}, \alpha_{ij}, v^s, \theta, w, v^w) )</td>
<td>( h = h(\pi_{ij}, \alpha_{ij}, p, s^w, w, s^w) )</td>
<td>( g = g(\pi_{ij}, \alpha_{ij}, p, \theta, w) )</td>
</tr>
<tr>
<td>( = u^s(\Delta_{ij}, \alpha_{ij}, v^s, s^w) + wu^w(v^w, s^w) )</td>
<td>( = f^s(\Delta_{ij}, \alpha_{ij}, v^s, \theta) + wf^w(v^w, \theta) )</td>
<td>( = h^s(\pi_{ij}, \alpha_{ij}, p, s^w) + wh^w(p, s^w) )</td>
<td>( = g^s(\pi_{ij}, \alpha_{ij}, p, \theta) + wg^w(p, \theta) )</td>
</tr>
<tr>
<td>( f = u - s\theta )</td>
<td>( f^s = u^s - s^w \theta )</td>
<td>( h = u - \pi_{ij}\Delta_{ij}/\rho_o + p/\rho )</td>
<td>( g = u - s\theta - \pi_{ij}\Delta_{ij}/\rho_o + p/\rho )</td>
</tr>
<tr>
<td>( f^w = u^w - s^w \theta )</td>
<td>( h^s = u^s - \pi_{ij}\Delta_{ij}/\rho_o + pv^s )</td>
<td>( h^w = u^w + pv^w )</td>
<td>( g^s = u^s - s^w \theta - \pi_{ij}\Delta_{ij}/\rho_o + pv^s )</td>
</tr>
<tr>
<td>( \pi_{ij} = \pi_{ij} + p^t \delta_{ij} = \rho_o \frac{\partial u^s}{\partial \Delta_{ij}} )</td>
<td>( \pi_{ij} = \pi_{ij} + p^t \delta_{ij} = \rho_o \frac{\partial f^s}{\partial \Delta_{ij}} )</td>
<td>( \Delta_{ij} = -\rho_o \frac{\partial h^s}{\partial \pi_{ij}} )</td>
<td>( \Delta_{ij} = -\rho_o \frac{\partial g^s}{\partial \pi_{ij}} )</td>
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<td>( \pi_{ij} = \pi_{ij} + p^t \delta_{ij} = \rho_o \frac{\partial f^s}{\partial \alpha_{ij}} )</td>
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</tr>
<tr>
<td>( p = -\frac{\partial u^s}{\partial v^s} = -\frac{\partial u^w}{\partial v^w} )</td>
<td>( p = -\frac{\partial f^s}{\partial v^s} = -\frac{\partial f^w}{\partial v^w} )</td>
<td>( \rho_o \frac{\partial h^s}{\partial p} ); ( \rho_o \frac{\partial h^w}{\partial p} )</td>
<td>( \rho_o \frac{\partial g^s}{\partial p} ); ( \rho_o \frac{\partial g^w}{\partial p} )</td>
</tr>
<tr>
<td>( \theta = \frac{\partial u^s}{\partial s^s} = \frac{\partial u^w}{\partial s^w} )</td>
<td>( \theta = \frac{\partial f^s}{\partial \theta} ); ( \theta = \frac{\partial f^w}{\partial \theta} )</td>
<td>( \theta = \frac{\partial h^s}{\partial s^s} = \frac{\partial h^w}{\partial s^w} )</td>
<td>( \theta = \frac{\partial g^s}{\partial \theta} ); ( \theta = \frac{\partial g^w}{\partial \theta} )</td>
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