The Bearing Capacity of a Strip Footing on Clay under Combined Loading

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**ABSTRACT**

In this Paper we seek closed form solutions for the problem of failure of a strip foundation on undrained clay subjected to combined vertical, moment and horizontal loading. The motivation for the study comes from the offshore oil and gas industry, where the understanding of problems of foundations under general loading is important. The contact between the foundation and the clay is unable to sustain tension, and this feature of the problem considerably complicates the solution. It is found that the Bound Theorems of plasticity theory are insufficient to solve the problem, and a solution can only be found by adopting certain further hypotheses. Although the existence of an exact solution cannot be proven, apparent upper and lower bounds to a plausible solution are obtained. In order to derive these solutions, a new scaling procedure is introduced, which plays the same role for upper bounds as the effective width method for lower bounds for footing collapse loads.

**INTRODUCTION**

The problem of the capacity of foundations under combined loadings is of considerable importance in soil mechanics. In the offshore oil and gas industry, in particular, foundations are subjected to horizontal loads due to wind and wave forces. Several types of offshore foundation are essentially shallow footings (for example the spudcan footings of jack-up units, mudmats for fixed jackets, concrete gravity bases and the caisson foundations that have recently been developed). Whilst these footings are usually close to circular in plan, an essential precursor to the understanding of the behaviour of circular footings is the study of the simpler problem of strip footings. Of the latter, the simplest problem is that studied here: the strip footing on uniform clay.

The calculation of the vertical bearing capacity of a strip footing on clay is one of the classical problems in soil mechanics. The exact solution for the load per unit length of $V = (2 + \tau)cB$, where $c$ is the undrained shear strength of the soil in plane strain and $B$ is the footing width, is well-known and is proven using the Lower and Upper Bound Theorems of plasticity theory. The extension of this solution to general cases of vertical, moment and horizontal loading appears at first sight to be straightforward, but turns out to be a problem of considerable subtlety. Surprisingly it is found that no exact solution can be found, and indeed the existence of a unique solution is unproven. The purpose of this paper is to set out a methodology for obtaining a plausible solution, based partly on the Bound Theorems of plasticity, but of necessity also involving some *ad hoc* assumptions. Solutions are presented which bracket closely what is regarded as a “working” solution. The treatment of this apparently simple

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problem reveals some important lessons about the application of plasticity theory in soil mechanics.

The sign convention for loads follows the recommendations of Butterfield et al. (1997) and is shown in Figure 1, where \( V, M, H \) are the vertical force, moment and horizontal force respectively. The displacements corresponding to \( (V, M, H) \) are \((w, \theta, u)\), so that the increment of work done on the footing is \( dW = Vdw + Md\theta + Hd\)u. All displacements are assumed small. Application of loads is assumed to be sufficiently fast for undrained conditions to be considered, therefore the soil can be treated as a purely cohesive material which yields according to Tresca's criterion \( |\sigma_{\text{max}} - \sigma_{\text{min}}| = 2c \) where \( c \) is the undrained shear strength. The contact between the footing and the soil is assumed to be perfectly rough, but with no tensile strength.

The objective is to determine the shape of the failure surface in \((V, M, H)\) space, which in principle could be written as a function \( f(V, M, H) = 0 \). It is clear from the symmetry of the problem that \( f(V, -M, -H) = f(V, M, H) \), so that in the following we only consider solutions for \( M \geq 0 \), and the complete solution can be obtained by symmetry. Note, however, that there is no physical reason why \( f(V, M, -H) = f(V, M, H) \), although some of the solutions developed will also include this form of symmetry.

The solutions will be expressed in terms of normalised loads \( v = \frac{V}{CB}, \quad m = \frac{M}{CB^2} \) and \( h = \frac{H}{CB} \).

**PREVIOUS WORK**

The solution for the problem of combined vertical and horizontal loading (but with no moment) is due to Green (1954). The problem of vertical and moment loading (but with no horizontal load) is usually considered by applying the "effective width" concept introduced by Meyerhof (1953). Both these approaches are discussed in more detail below.

The general case of vertical, horizontal and moment loading has received less attention. Although several authors (notably Meyerhof (1953), Hansen (1970) and Vesic (1975)) provide procedures for the general case, they do not examine it in detail, but instead adopt empirical
generalisations of the simpler cases. Ngo Tran (1996) compares the different solutions currently used.

Salençon and Pecker (1995a,b) made a valuable contribution to the analysis of the general problem, applying the Upper and Lower Bound Theorems of plasticity theory. The Authors’ view is that the problem is, however, more subtle than Salençon and Pecker suggest, and that this additional subtlety means that an exact solution to this apparently simple problem is unattainable. Instead we seek here what we will call a “working” solution. The points of difference between the approach presented here and the work of Salençon and Pecker are highlighted in the discussion of the results below.

Ngo Tran (1996) presents numerical solutions to the general problem, reporting both finite element analyses and use of an explicit finite difference code. The results are entirely consistent with the solutions developed here.

Bransby and Randolph (1997a,b) have studied the closely related problem in which full adhesion of the footing to the foundation is assumed. They derive a variety of numerical solutions based on finite element analysis and use of the Upper Bound Theorem, and include the effects of increasing soil strength with depth. The assumption of adhesion of the footing greatly simplifies the problem. To understand why the possibility of separation of the footing from the soil makes the problem so complex, it is necessary to explore some of the background of plasticity theory, and this is done in the next section.

NORMALITY, UNIQUENESS, CONVEXITY AND THE BOUND THEOREMS OF PLASTICITY THEORY

Much of the following discussion depends on the very important issues of the convexity of yield surfaces, and of the normality of the conjugate strain or displacement vectors. It is useful therefore to summarise some of the important consequences of these assumptions.

Firstly, if an elastic-plastic material obeys the stability hypothesis of Drucker (1959), then it is well known that:

(a.1) The yield surface in stress space is convex.² ³
(a.2) The plastic strain increment vector is normal to the yield surface (when strain increments are plotted on the same axes as the conjugate stresses); in other words the plastic potential is the same function as the yield surface.

The convexity and normality assumptions may of course be adopted independently, but are usually treated as consequences of Drucker’s hypothesis. In the following we restrict attention to elastic-perfectly plastic materials, i.e. excluding strain hardening or softening, since only the perfectly plastic case is relevant to the foundation problem addressed here.

² ³ This must be modified slightly if elastic-plastic coupling (the modification of elastic properties during plastic deformation) occurs. In these cases minor concavities can occur in the yield surface (Palmer et al, 1967), but such cases are not relevant here.
If a mechanical system consists entirely of elastic-plastic elements each of which satisfies the normality and convexity conditions, then several important conclusions follow for the loads applied to that system:

(b.1) A unique failure surface exists in the space of the applied loads.
(b.2) The above failure surface is convex.
(b.3) Plastic displacement vectors at failure (when plotted on the same axes as the conjugate loads) are normal to the failure surface.
(b.4) Use of the Lower Bound Theorem (in brief, a solution that satisfies equilibrium and does not violate the yield criterion) results in a lower bound to failure loads.
(b.5) Use of the Upper Bound Theorem (in brief, a plastic work calculation based on a compatible mechanism) results in an upper bound to the failure loads.

It is vitally important to realise that all five of the above consequences depend on the fact that the system consists entirely of elements that satisfy convexity and normality. If any component of the system violates this condition then all of the above consequences are lost – possibly with devastating results. Some of the unpleasant conditions that arise in this case are:

(c.1) A unique failure surface can no longer be defined. (This may appear to defy the common-sense assumption that the failure surface should be unique, but such an assumption is over-optimistic. It is quite reasonable that the failure surface should, for instance, depend on the history of loading).
(c.2) No statement can be made about the convexity or otherwise of the (non-unique) failure surface.
(c.3) No statement can be made about normality of the plastic displacement vectors to the (non-unique) failure surface.
(c.4) Neither the Upper nor Lower Bound theorems are valid (although note that Palmer (1965) derived some greatly weakened theorems for frictional materials).
(c.5) Apparent “Upper Bounds” may be calculated which are lower than apparent “Lower Bounds” (Drucker, 1954).

What is the relevance of all this to the footing problem studied here? The important issue is that at the interface between the footing and the soil the normality condition is violated. “No-tension” interfaces such as this show behaviour superficially similar to plasticity rules, but their behaviour is different in important respects.

Consider a horizontal interface in which the tractions acting on the lower half are as shown in Figure 2(a) and the corresponding displacements of the upper with respect to the lower half are as in Figure 2(b). The “yield surface” for this interface is the line AOB (produced to infinity in both directions) in Figure 2(c). All states below this line (negative $\sigma$: we are using here a tensile positive convention) are accessible, and all states above are inaccessible.

At the instant that “yield” occurs, i.e. contact is lost, the behaviour is not like that of a conventional plasticity model. Since loss of contact involves both $\sigma = 0$ and $\tau = 0$, the stress point immediately jumps to point O, irrespective of the position of the stress point (anywhere along AOB) immediately before “yield”. 
Figure 2: Behaviour of a no-tension interface

At the instant that separation occurs the velocity vectors must involve a positive component \( dv \), but \( du \) is undetermined, \( i.e. \) the motion may include a tangential component. Possible directions of the velocity vector are shown by the arrows radiating from point O on Figure 2(c). This behaviour can be understood as a limiting case of a non-dilative frictional interface as the angle of friction approaches \( 90^\circ \).

The above behaviour should be contrasted with that given by a plasticity model with the same yield locus but an associated flow rule (\( i.e. \) obeying the normality condition, see Figure 2(d)). In this case yield is possible anywhere along the line \( \sigma = 0 \) (no jump of stresses to the point O is necessary) and the displacement vectors contain no component in the \( u \)-direction – \( i.e. \) separation is not accompanied by any tangential movement. This plasticity model has the same “yield locus” as the model for the no-tension interface, but because normality is introduced, the physical behaviour that it represents is quite unreasonable. This is because (a) shear stress can apparently be sustained immediately after separation and (b) the separation is never accompanied by tangential movement.

It can be seen therefore that a plasticity model with normality cannot properly describe a no-tension interface: a more subtle physical response occurs in reality. The presence of the no-tension interface in the analysis of the footing under combined loads means that (unfortunately) all the benefits (b.1) to (b.5) listed above disappear, and all the problems (c.1) to (c.5) may occur! The important points of difference between the present study and the work of Salençon and Pecker are that they make use of various of the benefits (b.1) to (b.5). Salençon and Pecker (1995a) effectively analyse the case in which the behaviour of the
interface is as defined in Figure 2(d), but as that does not correspond to any physically reasonable system, the applicability of the solutions is questionable.

The undesirable effects (c.1) to (c.5) above are not simply academic points of little relevance to reality. When the more realistic behaviour of the interface as defined in Figure 2(c) is used, it is, for instance, easy to derive apparent “Upper Bounds” which are lower than apparent “Lower Bounds”. A simple example is given in the Appendix. Clearly a single case of this sort is sufficient to invalidate the approach based solely on the Bound Theorems.

How can one proceed with the analysis of such a problem? At present the only feasible analytical approach requires that use of the plasticity theorems must be tempered by a certain degree of engineering judgement, in that certain apparent “Upper Bound” and “Lower Bound” solutions have to be rejected on the grounds that they lead to unreasonable results. Although the existence of a unique exact solution cannot be proven, we will proceed as if such a solution does exist and will attempt to seek it.

SELECTION OF SOLUTIONS

In order to proceed it is necessary to supplement the Bound Theorems with two working hypotheses. These are introduced as required below. The first is:

Hypothesis 1: If any solution can be found for which (a) the Upper Bound Theorem is satisfied, (b) the Lower Bound Theorem is satisfied, (c) the Upper Bound solution does not involve contact-breaking at the soil-footing interface and (d) the Lower Bound solution does not involve tensile stress on the soil-footing interface, then this will be assumed to be part of the “working” solution.

Any solution satisfying the first three of these conditions would be exact for the case of a fully adhesive interface, and Drucker (1954) shows that this solution will be an Upper Bound to the problem with a frictional interface. If this solution also satisfies the fourth condition then it seems entirely plausible that it can be treated as a “working” solution for a no-tension interface.

The solution due to Green (1954) (see also Bolton (1979), pp 323-324) for combined vertical and horizontal loading (but \( M = 0 \)) satisfies Hypothesis 1. The Lower Bound stress field is as shown in Figure 3. The stress field consists of two wedges (1) and (3) each of homogeneous stress, separated by a fan shear zone (2). It is usually assumed that this stress field can be extended to the infinite half space without violating equilibrium and the yield criterion. However, a formal proof of this is not known and is not trivial because of the asymmetry of the problem. Without such an extension the solution is not a rigorous Lower Bound. The extension of the stress field in the immediate vicinity of the corners of the footing is straightforward. It seems highly likely therefore that with sufficient ingenuity an appropriate extension could be found, and this will be assumed here to be possible.

Salençon and Pecker (1995a) present a rigorous Lower Bound (for an adhesive foundation), including the extension to the full half-space, which is identical to the Green (1954) solution for most of the range of vertical load values, and falls slightly within it for a small part of the range at high vertical loads. The Green solution will, however, be accepted in the following.
Simple analysis of the stress state below the footings using conventional plasticity theory (Green, 1954) gives the following expression for the dimensionless forces:

\[ v = 1 + \frac{\pi}{2} + \cos^{-1} \left( h + \sqrt{1 - h^2} \right), \quad m = 0 \tag{1} \]

The corresponding Upper Bound involves zones (1) and (3) moving as rigid blocks, with distortion of zone (2), and yields the same solution for the loads.

When \( \theta = \pi / 4 \) the solution degenerates to the case of horizontal sliding at the interface, for which one can show:

\[ 0 \leq v \leq 1 + \frac{\pi}{2}, \quad h = \pm 1, \quad m = 0 \tag{2} \]

Equation (1) is a cycloidal arc that is tangential to the straight line of equation (2). The solution, which is symmetrical about the \( v \)-axis, is shown for \( h \geq 0 \) in Figure 4, in which AB is the solution represented by equation (2) and BC the solution from equation (1).
AN APPARENT LOWER BOUND FOR THE GENERAL CASE

Most engineers find Lower Bound solutions (i.e. those which satisfy equilibrium and nowhere violate the yield criterion) intuitively appealing, and it seems reasonable to continue to use the same principles to obtain solutions even for cases where the Lower Bound Theorem does not strictly apply. These solutions will be termed “Apparent Lower Bounds” (similarly the term Apparent Upper Bound will be used when Upper Bound principles do not strictly apply). A note of caution should perhaps be introduced, that the stress fields should be restricted to those that might reasonably be considered as similar to those that might actually occur in the real problem. In other words highly contrived and unlikely stress fields should be excluded. Note that Drucker (1954) resorted to a rather unlikely stress field (see his Figure 4) to demonstrate (for a case involving non-associated flow) an “Apparent Lower Bound” indicating stability when an “Apparent Upper Bound” indicated instability.

The “Apparent Lower Bound” surface obtained from the application of the effective width principle (Meyerhof, 1953) to Green’s solution on the \( M = 0 \) plane seems an entirely reasonable approach, although because of the interface behaviour it cannot be claimed as a rigorous Lower Bound. The effective width method is equivalent to the assumption that a solution for loads on a footing of width \( B_0 \) is also applicable to a footing of larger width \( B \), see Figure 5, in which the loads on the two footings are statically equivalent. If \( x = B_0 / B \), where \( 0 < x \leq 1 \), and the original solution is \( (v_0, m_0, h_0) \), then the derived solution is \( (v, m, h) = \left( x v_0, x^2 m_0 + \frac{1}{2} x(1-x)v_0, x h_0 \right) \), where the upper sign is for extension of the
footing to the right and the lower sign is for extension to the left (as in Figure 5). In most cases the effective width concept is applied to solutions with \( m_0 = 0 \). It is convenient to eliminate \( x \) to give:

\[
\frac{h}{v} = \frac{h_0}{v_0}, \quad \frac{v \pm 2m}{\nu^2} = \frac{v_0 \pm 2m_0}{\nu_0^2}
\]  

(3a,b)

Applying this procedure to the Green (1954) solution we make use of \( m_0 = 0 \) obtain the Apparent Lower Bound surface (for positive \( m \), which corresponds to the extension to the left):

\[
\frac{\nu^2}{\nu - 2m} = 1 + \frac{\pi}{2} + \cos^{-1}\left( \frac{hv}{\nu - 2m} \right) + \sqrt{1 - \left( \frac{hv}{\nu - 2m} \right)^2}
\]  

(4)

from equation (1) and:

\[
\frac{hv}{\nu - 2m} = 1
\]  

(5)

from equation (2).

The Apparent Lower Bound surface can be presented as a set of contours of constant normalised moment in the \((v, h)\) plane, Figure 6. The outermost contour corresponds to \( m = 0 \) and consists of the cycloidal arc and straight line from Green’s solution. The contour interval is 1/16. The surface is symmetrical with respect to both the \( m = 0 \) and \( h = 0 \) planes and has two areas of concavity at low \( v \) and high \(|h|\). A feature of this surface is that any section of it by a plane containing \( m \)-axis \((i.e. \ h/v = \text{constant})\) is a parabola.
APPARENT UPPER BOUND SOLUTIONS FOR THE GENERAL CASE

General approach

When the Apparent Upper Bound solution is constructed, there are two cases that must be treated in different ways:

- Case I - no contact breaking occurs between the footing and the soil;
- Case II - contact breaking occurs.

The first case is solved by analysing velocity fields only for those kinematic mechanisms that do not involve contact breaking. Each mechanism produces a surface in \((v, m, h)\) space. These surfaces intersect each other, and we are interested in the lowest possible combination of these surfaces. The second case is solved using (a) solutions from Case I for the kinematic mechanisms that are at the onset of contact breaking and (b) a special procedure of scaling the footing width, as described below. When these procedures are used, no Apparent Upper Bounds are found to fall below Apparent Lower Bounds.

The Scaling Concept

If a straight line is drawn tangential to the base of the displaced footing, it will generally be found that this line passes through part of the displaced soil to either side of the footing. Most of the mechanisms considered below include a limiting case, however, when at one side of the footing this no longer occurs. This limiting case can be considered as an onset of contact breaking. If \(w_s(x)\) is the vertical displacement of any point on the soil surface, then this kinematic condition is expressed as:

\[
w_s(x) \geq w + \theta x B
\]

(6)

Where \(w\) and \(\theta\) are the vertical displacement and rotation of the footing and \(xB\) is the horizontal co-ordinate measured from the footing centreline. If inequality (6) is satisfied for \(-\infty < x \leq -B/2\) then the footing could be extended by an imaginary section to the left without the extension making contact with the soil. Alternatively if it is satisfied for \(B/2 \leq x < \infty\) then the footing could be extended to the right. The concept is illustrated in Figure 7, where the mechanism is scalable by extension to the left since the deformed soil surface lies entirely below the projection AB of footing base. The illustrated mechanism is clearly not scalable by extension to the right. Extension to the left will only be possible when \(\theta\) is positive and the extension to the right only when \(\theta\) is negative. These extensions are described as “scaled” mechanisms, and the scaling concept is the exact equivalent for

![Figure 7: The scaling concept](image-url)
Figure 8: A section of the “exact” solution surface

Apparent Upper Bounds of the effective width concept for Apparent Lower Bounds. Mechanisms for which scaling by extension is possible will be called scaleable. Scaling in which the footing width is reduced is always allowable since no kinematic conditions prevent it.

The scaling cases that need to be considered can be reduced, since it can be shown that $\theta$ must have the same sign as $m$ if the following hypothesis is adopted:

Hypothesis 2: The Plastic Potential surface for the “working” solution is convex in the $(m,h)$ plane.

Consider a section of the (as yet unknown) “working” yield surface at constant $\nu$, (Figure 8). There are two points on this surface for which we already have a solution that satisfies Hypothesis 1 and can therefore be treated as part of the working solution. These are points A and B, both at $m = 0$, on this surface. Since the Apparent Upper Bounds for these points involve no rotation it follows that the direction of the plastic potential is also known at these points, giving flow in the directions shown. From Hypothesis 2, it follows that around the plastic potential surface the direction of incremental displacement vector $\tan^{-1}(B\theta/\nu)$ is a monotonically varying function of the direction of the force vector $\tan^{-1}(m/h)$. Therefore, (since $\theta = 0$ at $m = 0$) the sign of the footing rotation is the same as the sign of the applied moment. The assumption that $m$ and $\theta$ have the same sign is in fact a weaker assumption than Hypothesis 2, and only this weaker assumption is necessary in the following. The Authors consider that convexity of the plastic potential is a weaker assumption than convexity of a yield surface, and is more defensible on physical grounds. (i.e. that an increase in the ratio $m/h$ always results in an increase in the ratio $Bd\theta/\nu$).

Since application of a positive moment $m$ will lead to a positive rotation $\theta$, scaling by extension to the left need only be considered.

The surfaces for Case II, which involve contact breaking, are produced by the above scaling procedure. If the combination of loads $(V_o, M_o, H_o)$ represents an Apparent Upper Bound
estimate of the bearing capacity of the footing of width \( B_o \) at the onset of contact breaking, it will also represent an Apparent Upper Bound estimate for the extended footing of width \( B \) with appropriate adjustment of the moment to account for the position of the footing centreline (see Figure 5). This leads to equations for the scaled surfaces that are identical to those obtained using the effective width concept (equations 3a,b), with the lower signs applying for the extension of the footing to the left. The scaled surfaces produce parabolas when intersected by the planes including \( m \)-axis, in a similar way to the Apparent Lower Bound surface, obtained using the effective width concept.

Each kinematic mechanism is represented by a family of planes in \((v, m, h)\) space. Consider a section of this space by a plane \( h/v = constant \). Figure 9 shows projections of this section onto the \( h = 0 \) plane. A particular mechanism results in a line that is the intersection of its particular plane with the section \( h/v = constant \), and the projection of this line onto the \( h = 0 \) plane is the line AB in Figure 9. Since it was shown from Hypothesis 2 that the footing rotation has the same sign as the moment, it can be shown that AB has a negative slope for positive \( m \), and the extension of this line below its intersection with the \( m = 0 \) plane (Point B) can be disregarded.
Consider a point C on AB, which represents a possible combination of normalised loads \((v_o, m_o, h_o)\) corresponding to this particular mechanism. If AB represents a scaleable mechanism then the scaling of the point C produces a parabola (equation 3b) which is the envelope of the scaled mechanisms. The parabola passes through the origin, point C by definition and point D on \(m = 0\), i.e. it is the curve OCD in Figure 9(b). The slope of the parabola at the origin is 0.5. The part of the parabola OC between the origin and the point C is produced by extension, while the part CD corresponds to reduction of the footing's width.

Even if the mechanism is not scaleable by extension, it can still be scaled by reduction so that the part of the parabola CD can always be used. The envelope of possible mechanisms in this case is the curve ACD in Figure 9(a).

Since we are seeking an Apparent Upper Bound solution, we are interested in the lowest possible parabola. Clearly this parabola is the one that is tangent to the line AB, so that C has been chosen on AB such that the parabola is tangential to AB at C. If the absolute value of the slope of AB is greater than 0.5 (Figure 9(c)), the tangential point C would fall in the range of negative \(m\), which contradicts Hypothesis 2. In this case, the lowest parabola should just
pass through the point B (Figure 9(d)). It can be shown, however, that these cases correspond to mechanisms in which the instantaneous centre falls outside the left hand side of the footing, and such mechanisms are highly unlikely to be scaleable. Possible Apparent Upper Bounds are presented by thick lines for non-scaleable mechanisms in Figs. 9(a) and 9(c) and for scaleable mechanisms in Figures 9(b) and 9(d).

The mechanisms without contact breaking generate a family of planes that form an envelope (which is possibly not smooth). This envelope is bounded by a curve corresponding to the mechanisms at the onset of contact breaking, and the scaling procedure is applied to points on this curve. For the particular mechanisms used in these solutions (see below) there is a smooth transition between the surfaces produced by non-scaleable mechanisms (Case I) and the surfaces produced by scaling of the bounding curve (Case II). In other words the bounding curve cuts the section in Figure 9(b) at point C. Furthermore, it was observed that the branches of scaling parabolas obtained by reduction of the footing’s width (CD in Figure 9(b)) do not lie lower than the envelope of non-scaleable mechanisms. Hence, the only scaling which contributes to the Apparent Upper Bound construction is the scaling represented by the parabola OC in Figure 9(b).

**Kinematic Mechanisms**

**Mechanism I: Slip along the Contact Surface**

This mechanism produces an Apparent Upper Bound in the form of two planes \( h = \pm 1 \) as already presented (equation (2)). The onset of the contact breaking is given by two straight lines parallel to the \( v \)-axis that can then be used for scaling:

\[
h_o = \pm 1, \quad m_o = 0
\]  

(7)
Mechanism 2: Green’s solution

The Apparent Upper Bound mechanism shown in Figure 3 was developed for central loading, but is equally applicable when a moment is present. The surface in \((v, m, h)\) space is given by equation (1). This mechanism is not scaleable because it involves no rotation.

Mechanism 3: Pure Rotation

This mechanism involves pure rotation of a rigid block around a centre \(\Omega\) (Figure 10). The work equation leads to the following expression representing a set of planes in \((v, m, h)\) space depending on two parameters: the \((x, y)\) co-ordinates of the centre.

\[
-vx - hy + m = 2\beta \left( y^2 + (0.5 - x)^2 \right)
\]  

(8)

where \(\tan \beta = \frac{0.5 - x}{y}\).

This set of planes produces in part an envelope that is found by differentiating the work equation with respect to \(x\) and \(y\):

\[
\frac{h}{v} = \frac{\tan \beta - 2\beta}{1 + 2\beta \tan \beta}, \quad \frac{v - 2m}{v^2 + h^2} = \frac{\beta}{1 + 4\beta^2} \quad (0.5 \leq \beta < \pi / 2)
\]  

(9)

The onset of contact breaking is represented by the case when the centre of rotation lies on the centreline of the footing \((x = 0)\). In this case normalised loads are found from the following equations, representing a curve, bounding the above envelope:
By scaling this curve using equations (3), we extend the surface into the contact breaking zone of the \((v, m, h)\) space. In fact, for this particular mechanism, the envelope of mechanisms without contact breaking can also be obtained by scaling the \(x = 0\) solution by reduction of footing width, leading to the same expressions (equation 9).

**Mechanism 4: Hansen’s mechanism**

This mechanism, see Figure 11, was proposed by Brinch Hansen, and the analysis of the undrained case is given by Salençon and Pecker (1995a). It involves rotation of the rigid block around a centre \(\Omega\) and distortion of the fan and wedge zones. A limiting case of this mechanism occurs when the centre of rotation \(\Omega\) is at infinity, and it reduces to the basic combined mechanism, corresponding to the stress field of the Green’s Lower Bound solution.

For normalised loads, the following work equation is obtained, depending on three parameters: the \((x, y)\) co-ordinates of the centre and an angle of the wedge \(\mu\):

\[
-vx - hy + m = -x \left( \pi - \mu + \frac{\tan \mu}{2} \right) + y^2 \left( \frac{\beta}{\sin^2 \beta} - \frac{\alpha}{\sin^2 \alpha} \right)
\]  

(11)

where \(\tan \alpha = \frac{y}{0.5 - x}\), \(\tan \beta = \frac{y}{x + 0.5}\)

This set of planes produces an envelope and its equations are found by differentiating the energy equation with respect to \(x\), \(y\) and \(\mu\). The optimal value of \(\mu\) is \(\pi / 4\), and the resulting surface is most conveniently expressed in parametric form in terms of \(\alpha\) and \(\beta\):

\[
\frac{2m}{h + 1} = \frac{1}{\beta - \alpha} - \frac{1}{\tan(\beta - \alpha)}
\]  

(12)
Figure 12: Green’s mechanism with rotation

\[
\begin{align*}
\nu + 2\alpha &= \frac{h + 1}{\tan \beta} + \frac{3\pi}{2} + 1 \\
\nu + 2\beta &= \frac{h + 1}{\tan \alpha} + \frac{3\pi}{2} + 1
\end{align*}
\] (13) (14)

For the special case, when \( \alpha = \beta \), the centre of rotation is at infinity, \( m \) is equal to zero and the yield surface is given by the equation (1).

The onset of contact breaking is represented by the case when the centre of rotation lies below the corner of the footing (\( x = -0.5 \)). For this case \( \beta = 0 \) and the following normalised loads expressed parametrically in terms of \( \alpha \) form the bounding curve to be used in the scaling procedure:

\[
\nu_o = \frac{3\pi}{2} + 1 - 2\alpha, \quad h_o = (\pi - 2\alpha)\tan \alpha - 1, \quad m_o = \tan \alpha + \alpha - \pi/2
\] (15a,b,c)

Mechanism 5: Green’s Solution with Rotation

This mechanism is the basic combined mechanism with addition of rotation of the centre of the footing around a centre \( \Omega \) lying on the x-axis, see Figure 12. When the centre of rotation is at infinity this mechanism also reduces to Mechanism 2. The work equation depends on two parameters: the co-ordinate of the centre \( x \) and an angle of the wedge \( \beta \):

\[
\nu + h\cot \beta - \frac{m}{x} = 1 + \frac{\pi}{2} + \left(1 - \frac{1}{4x}\right)\cot \beta + 2\beta
\] (16)

To obtain the envelope of these planes we first differentiate with respect to \( \beta \), resulting in:

\[
\nu + 4\xi m = 1 + \frac{\pi}{2} + \cos^{-1}(h - \xi) + \sqrt{1 - (h - \xi)^2}
\] (17)

where \( \xi = -1/(4x) \). The envelope of this set surfaces is not itself a smooth surface, but instead it is bounded by two ruled surfaces corresponding to the limiting values of \( \xi \). The first corresponds to \( \xi = 0 \) (i.e. \( x \to -\infty \)), it is formed from lines parallel to the m-axis and its formula is the same as equation (1). The second surface corresponds to \( \xi = 0.5 \) (i.e. \( x = -0.5 \),
rotation about the edge of the footing) and is ruled by a set of lines \( v + 2m = f(h) \). In fact it is found later that this second surface never proves to be a critical mechanism.

**Mechanism 6: Rotation with Distortion**

This mechanism involves rotation of the rigid block around a centre \( \Omega \) and distortion of the fan and wedge zones (Figure 13). The limiting cases of this mechanisms are when the centre of rotation \( \Omega \) is at infinity and it reduces to Mechanism 2, corresponding to the stress field of the Apparent Lower Bound solution, while when the centre is placed at the symmetry axis it reduces to Mechanism 3. For normalised loads, the work equation can be obtained, and it represents a set of planes depending on three parameters: the \((x,y)\) co-ordinates of the centre and an angle of the wedge \( \mu \):

\[
-\nu x - hy + m = -x \left( \pi - \mu + \frac{\tan \mu}{2} \right) + y^2 \left( \frac{\beta}{\sin^2 \beta} - \frac{\alpha}{\sin^2 \alpha} \right) + y \tan \mu \left( \frac{1}{\sin \beta} - \frac{1}{\sin \alpha} \right)^2
\]  

(18)

where \( \tan \alpha = \frac{y}{0.5 - x} \), \( \tan \beta = \frac{-y}{x + 0.5} \)

To obtain the envelope of these planes we first differentiate with respect to \( \mu \), resulting in:

\[
\cos 2\mu = \frac{2(\sin \beta - \sin \alpha)}{\sin \beta + \sin \alpha}
\]

(19)

The envelope of the set of planes (18) is not itself a smooth surface, but instead it is formed by two smooth surfaces. One of them is the same vertical wall as in Mechanism 5: it is formed from lines parallel to the \( m \)-axis and its formula is the same as equation (1). On the second surface the onset of contact breaking is represented by the case when the centre of rotation lies on the centreline of the footing \((x = 0)\). In this case normalised loads are found from
equations (10), representing a curve, bounding the above envelope. It is found that the second surface never proves to be a critical mechanism.

**Apparent Upper Bound surface**

After examining the above mechanisms, it is possible to find the smallest Apparent Upper Bound surface. This is presented as a set of contours of constant normalised moment in the \((v,h)\) plane (see Figure 14). The outermost contour corresponds to \(m = 0\) and coincides with the Apparent Lower Bound solution. The step between the contours is 1/16.

The surface obtained is a combination of the following zones:

- **Zone A** - Mechanism 1: slip along the contact surface plus scaling;
- **Zone B** - Mechanism 3: pure rotation plus scaling;
- **Zone C** - Mechanism 4: Hansen's mechanism plus scaling;
- **Zone D** - Mechanism 2/5/6: vertical wall from Green's solution.

The surface is rotationally symmetrical with regard to the \(m = 0\) plane and, like the Apparent Lower Bound surface, has concavities at low \(v\) and high \(|h|\) values.

**Numerical Procedures for Apparent Upper Bounds**

The objective of this study was to demonstrate the difficulties in the search for Apparent Upper Bound solutions when contact breaking is involved, and to introduce the scaling technique as a simple engineering tool to develop these solutions. This has been achieved by analysing those mechanisms that allow closed-form solutions of the problem.

Numerical analysis of the problem could allow consideration of a much wider range of mechanisms. The numerical procedure would involve at each stage a choice of the parameters
defining a particular mechanism. Each of the mechanisms will correspond to one of the four cases in Figure 8, and the envelope of all mechanisms can be cut by the appropriate surface. The lowest possible combination of these surfaces would form an Apparent Upper Bound.

Such a procedure would allow examination of mechanisms which involve large numbers of defining parameters, for instance the mechanism shown in Figure 15, which was considered by Bransby and Randolph (1997a,b). This mechanism involves rotation of the rigid block around a centre $\Omega$ and distortion of the fan and two wedge zones. The limiting cases of this mechanism are when the centre of rotation $\Omega$ is at infinity, when it reduces to the basic combined mechanism, corresponding to the stress field of the Apparent Lower Bound solution, and when the centre is placed at the symmetry axis, when it reduces to the pure rotation mechanism. The work equation depends on five parameters: two co-ordinates of the centre $x$ and $y$, the radius $r$, two angles of the wedges.

Although the Authors believe that other mechanisms are unlikely to lower the Apparent Upper Bound significantly, introduction of a numerical procedure should stimulate a search for new scaleable and non-scaleable mechanisms.

**DISCUSSION OF THE RESULTS**

Both Apparent Lower and Apparent Upper Bound estimates are presented in a form of the contours of constant eccentricity in the $(v, h)$ plane (Figure 16). These contours are derived by intersecting the Apparent Lower and Apparent Upper Bound surfaces by planes containing $h$-axis. Apparent Lower and Apparent Upper Bound estimates coincide on the $m = 0$ plane and along the surfaces derived by scaling or effective width concept from solutions for slip along the surface. In these cases it is reasonable to assume the existence of unique working solutions. For the portions of the surfaces that do not coincide, the bound estimates give a reasonably narrow band for the possible location of the working yield surface.

This solution can be compared with one presented by Salençon and Pecker (1995a), Figure 17. The discrepancies between these two solutions can be explained as follows:

- Salençon and Pecker admitted some inaccuracy in numerical minimisation of equations since they did not use closed form solutions for the envelopes of Upper Bounds;
Figure 16: Solutions presented as contours at constant eccentricity

- Salençon and Pecker's Upper Bound appears to be much higher than it could be, since they do not use the scaling procedure for the contact breaking case;
- Salençon and Pecker's Lower Bound was pushed higher using a convexity argument, which is not valid in the contact breaking case.

The last of these points is probably the most important. Salençon and Pecker (1995a) assume that their Lower Bound surface should be convex, and so they extend the surface given by equations 4 and 5 by bridging across the concavities with a ruled surface. Since the convexity of the surface cannot be proven for this case in which normality is violated, this extension would appear to be somewhat dangerous. In fact the extension is rather small. The rather large effect which is apparent in Salençon and Pecker's Figure 10 is because of the choice of sections through the surface at constant eccentricity \( e = M/V \): this has the effect of exaggerating the significance of the extension. The difference between the bounds presented by Salençon and Pecker is greater than the difference obtained in this work, and in some regions the solutions derived here fall entirely below theirs.

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Figure 17: Salençon and Pecker’s solution presented as contours at constant eccentricity

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APPENDIX: An Apparent Upper Bound lower than an Apparent Lower Bound

As a simple example, consider an Apparent Upper Bound mechanism as shown in Figure A1.

![Figure A1: Upper Bound Mechanism](image)

Failure occurs on a circular arc of radius $R$ centred at a point $R \cos \theta$ above the footing and a distance $a$ to the left of the centreline. Loss of contact therefore occurs on a length $2a$ at the right hand edge of the footing. For an anticlockwise rotation $\delta$ the “Upper Bound Theorem” gives:

$$c. R \theta . R \delta \geq -Va \delta + HR \delta \cos \theta - M \delta$$  \hspace{1cm} (A1)
Making use of \( a = \frac{B}{2} - R \sin \theta \) this may be re-written

\[
2cR^2 \theta \geq -\frac{VB}{2} + VR \sin \theta + HR \cos \theta - M
\]  
(A2)

For simplicity consider a particular case of proportional loading in which \( V = 2H \) and \( M = 0 \), and we wish to place an upper bound on \( H \). To simplify the problem further, consider only the case \( R \sin \theta = 0.3B \) (i.e. \( a = 0.2B \)). The above inequality becomes

\[
2c(0.3B)^2 \frac{\theta}{\sin^2 \theta} \geq -0.2B(2H) + H(0.3B)\frac{\cos \theta}{\sin \theta}
\]  
(A3)

which is readily re-arranged as:

\[
H(0.3 - 0.4 \tan \theta) \leq 0.18cB \frac{2\theta}{\sin 2\theta}
\]  
(A4)

As \( \theta \) becomes small this gives the inequality \( H \leq 0.6cB \), so that the point \( (V, M, H) = (1.2cB, 0, 0.6cB) \) must lie on the Apparent Upper Bound surface. The corresponding solution which falls on the Apparent Lower Bound surface (equation 5) is \( (V, M, H) = (2cB, 0, cB) \), so that the Apparent Upper Bound for this case of proportional loading is considerably lower than the Apparent Lower Bound solution, amply demonstrating the breakdown of the bound theorem approach as soon as normality is violated by an element with the system. Clearly a single counter-example is sufficient to invalidate the use of the theorems without some modification, but the above could be extended to much more general cases.

In the main text the Bound Theorems have been supplemented by two further hypotheses, and these allow solutions to be obtained in which all Apparent Upper Bounds are above Apparent Lower Bounds.