The Plastic Response of Circular Footings on Sand under General Planar Loading

by

G. Gottardi\(^1\), G.T. Houlhsby\(^2\) and R. Butterfield\(^3\)

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University of Oxford,
Department of Engineering Science,
Parks Road, Oxford, OX1 3PJ, U.K.

Tel. 01865 273162/283300
Fax. 01865 283301
Email Civil@eng.ox.ac.uk
http://www-civil.eng.ox.ac.uk/

\(^1\) University of Bologna
\(^2\) University of Oxford
\(^3\) University of Southampton
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G. Gottardi¹, G.T. Houlsby², R. Butterfield³

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SYNOPSIS

Data from a set of tests of model circular footings on dense sand are presented. The footings were subjected to a variety of combinations of vertical, horizontal and moment loading. The tests were designed to provide the information necessary to construct a complete model of the footing behaviour, based on the concepts of plasticity theory. In particular the tests provide detailed information about the shape of the yield surface, and so allow generalisation of bearing capacity calculations to cases other than purely vertical loading. Information is also obtained about the hardening law and flow rule appropriate for a plasticity model, and the elastic response within the yield surface.

1. INTRODUCTION

Engineers need to be able to calculate the capacity of foundations subject to, at least, central vertical loads. This need led to the development of the theories of bearing capacity, notably Terzaghi's method. Although all standard bearing capacity analyses make some use of the concepts of plasticity theory, they all also employ empirical factors. In many instances, notably in offshore engineering and design of retaining structures, foundations have to support inclined and eccentric loads (i.e. horizontal loads \( H \) and moments \( M \) in addition to central vertical loads \( V \)), and further empirical factors have been added to the conventional analyses to allow for these complications. The best known of such procedures are those described by Meyerhof (1953), Brinch Hansen (1970) and Vesic (1973) although, until recently, little experimental data has been published which explores their validity throughout \((V, M, H)\) space in a systematic way.

Plasticity-based analyses may be constructed in terms of the force resultants acting on the footings and the corresponding footing displacements. Such an approach, which seems to have been first suggested by Roscoe and Schofield (1957), and more fully developed by Butterfield (1980, 1981), enables the concepts of vertical bearing capacity to be generalised to accommodate eccentric and inclined loads. This approach is regarded as a more powerful and consistent technique than the use of increasing numbers of empirical factors.

Early experimental studies of the behaviour of rigid foundations subjected to combined loads, expressed in the form of interaction diagrams, were published by Ticof (1977) and Butterfield

¹ University of Bologna
² University of Oxford
³ University of Southampton
and Ticof (1979). Recent interest in this area has led to more systematic work directed towards establishing the necessary components of plasticity models. Nova and Montrasio (1991) reported tests on strip footings on loose sand. Gottardi (1992) studied strip footings on dense sand. He used load-controlled tests (see also Gottardi and Butterfield (1993, 1995), Butterfield and Gottardi (1994)). Martin (1994) investigated circular footings (model spudcans) on clay, using a test rig with displacement control in which he carried out a comprehensive series of tests of both shallow and deep foundations in Kaolin clay, from which he developed a complete plasticity model. The general forms of Martin’s results for clay and Gottardi’s for dense sand were strikingly similar.

The object of this paper is to report the results of a comprehensive series of combined loading tests of circular footings on dense sand using a modified form of Martin’s displacement controlled apparatus. The tests were designed to provide the key data necessary for the construction of plasticity theories to model the foundation behaviour. The test programme complements Gottardi’s earlier results for strip footings on dense sand using load control and Martin’s work on clay, and comparison with these studies proves valuable.

Unfortunately the groups cited have used conflicting co-ordinate systems, sign conventions and notation. In an attempt to rectify this, the present authors recommended a unified and consistent system, Butterfield et al. (1997). This is the nomenclature used throughout this paper and the sign conventions for loads and displacements are shown in Figure 1. Note that the reference position for the loads is the current position of the centre of the base of the footing. The reference axes translate but do not rotate as deformation occurs. In order to describe loads and displacements in a dimensionally consistent way, loads are presented in terms of the three variables \((V, M/2R, H)\) (all with the dimension of force), and displacements – vertical, rotational and horizontal – in terms of \((w, 2R\theta, u)\) (all with the dimension of displacement).

2. EXPERIMENTAL EQUIPMENT

Soil Type

The soil used throughout the testing programme was dry 14/25 yellow Leighton-Buzzard sand. It is a very uniform sand (coefficient of uniformity \(CU = 1.3\)) with angular grain shape and specific gravity \(G_s = 2.66\). Particle diameter ranges from 0.6 mm to 1.18 mm \((D_{50} = 0.8\ mm)\). Maximum and minimum specific volumes are 1.79 and 1.49. The properties of this sand have been investigated by several researchers (see for example Stroud, 1974). At the low stress levels (typically less than 100 kPa) relevant to the tests in this series, the procedure suggested by Bolton (1986) for estimating the angle of friction gives for \(R_f = 75\%\) a peak angle of friction in triaxial compression of 42.3\(^\circ\), and in plane strain of 47.8\(^\circ\).

Sample Preparation

The testing tank was a cylinder made of dural, 450mm high, 450mm inside diameter and 6mm wall thickness, on the top of which the loading rig could be firmly attached. The tank was filled from a raining device with a slightly larger diameter than the tank. The rain of sand was cut by a cylindrical shield of the same diameter as the testing tank. Details of the raining
arrangements can be found in Gottardi and Houlsby (1995). The raining deposition technique provides very homogeneous and repeatable beds of dry sand, whose density can be varied by changing the intensity of flow of the sand. A target relative density of 75% was used throughout the testing programme. All the beds of sand used in the tests reported here were at relative densities in the range 73.9% to 75.9%.

After deposition the sand surface was accurately levelled using a vacuuming device. At the end of the test a layer of at least 50mm thickness (i.e. half the model footing diameter) was removed by vacuuming to achieve a new, essentially undisturbed, sand bed, enabling up to three tests to be performed on the same sample.

Experimental Rig and Data Acquisition

The model footing was of diameter \( D = 100 \text{mm} \), and thickness 7.38mm (chosen so that the load reference point at the base of the footag was conveniently placed with respect to the load cell position). A protective annular shield 100mm outside diameter and 30mm high was attached to the top of the footing. Sand was glued to the base of the footing to achieve a rough surface.

The experimental rig – completely automated and allowing independent control of the vertical, horizontal and rotational displacements of a model footing – was mainly designed and commissioned by Martin (1994), to investigate the response of spudcan footings to combined loading on soft clay. It was modified for use with dense sand specimens.

A general view of the rig fixed to the testing tank is given in Figure 2. The apparatus consists of a footing connected through a shaft to a system of plates whose movements can be independently controlled. Vertical \( (w) \), horizontal \( (u) \) and rotational \( (\theta) \) displacements of the footing are guided by separate bearing arrangements, two linear and one rotary, and driven by three stepper motors. In this way any desired footing displacement path in \( (w,2\theta,u) \) space can be achieved by superimposing the relevant displacement components. The shaft connecting the footing to the drive system incorporates a “Cambridge” type load cell (Bransby, 1973), which can measure the vertical \( (V) \), horizontal \( (H) \) and moment \( (M) \) loads acting on the footing. Such an apparatus is able to apply any planar displacement path to the footing and measure its response in terms of corresponding loads.

Additional instrumentation measures the three displacement components. Three long-travel displacement transducers are positioned to measure the movement of the three moving plates. A separate frame carrying a system of three short-travel \( (10\text{mm}) \) transducers is clamped to the testing tank to measure the displacements of the footing directly. This second set of displacement transducers allows accurate recording of small displacements, and avoids any effects of apparatus flexibility.

The tests were controlled and logged using a computer with suitable interfaces. The control program allowed sequences of events to be specified using an appropriate input file. Output data consisted of a time series of readings of all the transducers, typically at an interval of one second. Care was taken with appropriate calibration and zeroing of all transducers.
3. OBJECTIVES OF THE TESTS

Any experiments are most usefully interpreted in terms of a theoretical model, even if this is somewhat idealised. The hypothesis adopted here is that, after a given penetration of the footing, a yield surface will be established in \((V, M / 2R, H)\) space. Within this surface the displacements would be substantially elastic. If the footing is pushed further into the ground the yield surface expands. These concepts are described by Butterfield (1981), Houlsby and Martin (1992), Gottardi and Butterfield (1995).

The footings were subjected to a variety of displacement paths involving combined vertical translation, horizontal translation and rotation, and the corresponding loads were measured. In some tests feedback control was used to achieve control of loads rather than displacements. The forms of control were chosen to give information that enables plasticity models for the foundation behaviour to be derived.

Five broad classes of test were used: four of these are described and discussed in sections 4.1 to 4.4. Typical load paths for these four types of test are shown schematically in Figure 3, where it can be seen that the tests involve very different paths in load space. Elasticity tests, involving a series of small loading excursions \((V, M \text{ and } H)\) inside the yield locus, were used to estimate the elastic stiffness coefficients, but are not presented here.

Because the model tests reported here were carried out at 1g, they involve stresses significantly lower than those typically encountered in the field. The numerical values of the parameters derived from the tests at higher stress levels may be different from those found here, but the overall patterns of behaviour are expected to be similar.

4. TESTING PROGRAMME

The test programme consisted of 29 leading tests, the results of which are comprehensively described in Gottardi and Houlsby (1995). For each test a time-sequence record of load and displacement components is presented there, together with the load-displacement data in a variety of forms, including load-displacement plots, load paths and displacement paths. In many cases a specific part of the test is extracted, and separately plotted, to focus attention on the main events (e.g. the swipe phase, see below) carried out during the test. Only selected results leading to the principal conclusions are presented here. Therefore some of the 29 tests are not used (in particular, test GG01, which was at a slightly higher density than the rest, is omitted). For consistency, the numbering system for the tests is as used in Gottardi and Houlsby (1995).

All tests are grouped according to their type in Table 1 where information is also given about past loading and applied ratios of displacements. Typical displacement velocities were approximately 0.01 mm/s and 0.0002 rad/s. No specific investigation of rate effects, which were assumed to be negligible for dry sand, was carried out.
4.1. Vertical Loading Tests

**Vertical Loading Tests** involved displacement-controlled vertical loading and unloading. Their purpose was to establish the vertical load-displacement \((V,w)\) relationship up to the relevant maximum penetration, as well as information about the vertical elastic stiffness of the footing.

Four tests with displacement-controlled vertical loading and unloading were carried out (including test GG01). Figure 4 shows the load-displacement curves for two of these. Since the tests are displacement-controlled, the penetration can be continued after the peak load. Figure 3 provides the hardening rule for plastic vertical displacement, and also the elastic stiffness on unloading. In addition a further 15 tests began with a vertical loading phase up to 1600N, with an unload-reload cycle from 1000N: these tests provided good confirmation of the repeatability of the response up to 1600N.

A peak vertical load \(V_m\) in the range from 2000N to 2050N is indicated by the data, which is in good agreement with that estimated using the angle of friction for the sand, at the relevant relative density and stress level. The calculated value of \(\phi\) is approximately 255, which would imply, using the figures of Bolton and Lau (1993), an angle of friction of 41.3°. This is only 1° lower than the triaxial angle of friction estimated using the method of Bolton (1986). The range of densities measured in the tests would imply a variation in \(V_m\) not greater than about ±4%.

The unload-reload cycles show some non-linear behaviour, but are reasonably well fitted by an elastic stiffness \(k_e\) of about 20kN/mm.

In the interpretation of the swipe tests described later, an expression describing the \(V - w\) curve is needed. It proves to be most convenient first to divide the vertical displacement into elastic and plastic components \(w = w_e + w_p\). The elastic behaviour is then defined by:

\[
V = k_e w_e
\]  
...(1)

and the plastic response by a function of the kind \(V = f(w_p)\). An empirical expression, which has been found to represent the available experimental data well, is:

\[
V = \frac{k_p w_p}{1 + \left(\frac{k_p w_{pm}}{V_m} - 2\right)\left(\frac{w_p}{w_{pm}}\right) + \left(\frac{w_p}{w_{pm}}\right)^2}
\]  
...(2)

where \(k_p\) is the initial plastic stiffness, \(V_m\) is the maximum vertical load and \(w_{pm}\) is the value of \(w_p\) at this maximum load. The fit constructed from equations (1) and (2), with \(k_e = 20\) kN/mm, \(k_p = 2.0\) kN/mm, \(V_m = 2025\) N and \(w_{pm} = 3\) mm is shown on Figure 4. It is emphasised that no significance is attached to the particular form of equation (2), which is simply a convenient empirical expression that fits the hardening behaviour well.
Equation (2), with $V_0$ substituted for $V$, also serves later as a more general hardening relationship between the size of the yield surface and the plastic penetration.

### 4.2. Swipe Tests

Swipe Tests allow direct investigation of the shape of the yield surface at a given penetration. The procedure is as follows. A vertical displacement is applied until the footing reaches a specified load or penetration. At this stage the vertical drive is halted (i.e. $w$ is fixed) and, for example, the footing is driven purely horizontally. A horizontal load clearly develops, and usually some moment develops too. More importantly, whilst the horizontal load increases, the vertical load decreases, so that the load path sweeps out a track in $(V, M / 2R, H)$ space. It is argued (see for instance Tan (1990), Martin (1994)) that this path approximates closely to a track across the yield surface at the given penetration.

This type of test was first termed a “Sideswipe Test” by Tan (1990), since only horizontal displacement was used. Here the term “Swipe Test” is preferred since we also use it for tests in which pure rotation is applied rather than horizontal movement: in this case a moment is developed, usually accompanied by some horizontal load. It is also used for tests in which combinations of horizontal displacement and rotation are applied at fixed vertical displacement. By carrying out a systematic series of tests with different ratios of horizontal displacement to rotation, it is possible to build up a comprehensive picture of the shape of the yield surface. Figure 5 shows simplified views of the load paths followed over a generic three-dimensional yield surface for three possible swipe tests.

It is worth noting the striking parallel between this technique and the use of undrained triaxial testing to explore the shape of the State Boundary Surface for clays. The vertical load is the analogue of mean stress, vertical displacement that of specific volume (with a change of sign), and both horizontal force and moment are equivalent to shear stress. The combined loading problem therefore has an added dimension of complexity. Continuing this analogy, the tests described above are the equivalent of undrained tests on normally consolidated clay, and therefore give information about the shape of the yield surface at high vertical loads. To explore the yield surface at low vertical loads, tests analogous to overconsolidated triaxial tests must be used. This is achieved by loading vertically to a prescribed level, then unloading vertically to a different value before carrying out the swipe.

In detail the swipe tests do not follow the yield surface exactly, just as undrained triaxial tests do not follow the yield surface. The deviation proves, however, to be relatively small for the footing tests, and appropriate corrections (discussed below) can be made.

The swipe tests constitute the main body of the research programme since they provide most information about the shape of the yield locus. In total, 14 swipe tests were carried out. Six were from an initial vertical load of 1600N, which is about 80% of the peak load $V_n$. They all involved a first phase of pure vertical loading (with an unload-reload cycle at 1000N) and a second phase where the footing was moved at fixed vertical penetration. The only difference between the tests was the ratio between the applied components of rotation and horizontal displacement in the second part (see Table 1).
Figure 6(a) shows the load-displacement curve for the swipe phase for test GG03, when the footing is displaced purely horizontally at fixed vertical penetration. Figure 6(b) shows the corresponding load path, demonstrating that, as the horizontal load increases (and later decreases towards the end of the test), the vertical load continuously reduces. The load path followed is remarkably close to a parabola of the form:

\[
\frac{H}{H_o} = 4 \frac{V}{V_o} \left( 1 - \frac{V}{V_o} \right)
\]  

...(3)

where \(V_o\) is the starting vertical load and \(H_o\) is the maximum value of the horizontal load. If the elastic stiffness is high compared to the plastic stiffness, this load path corresponds closely to a track across the yield surface, so that the implication is that the yield surface has the same parabolic shape in the \((V, H)\) plane. The maximum value \(H_o = 0.13V_o\) is similar to the figure of \(0.12V_o\) found for the failure envelope for strip footings on sand (Ticof, 1977; Gottardi and Butterfield, 1993). A small moment was also developed as the test progressed.

Four swipe tests were also carried out to explore the shape of the yield surface at low vertical loads, starting from a point at which the foundation had first been loaded to 1600N and then unloaded to 200N. Figures 6(a) and 6(b) also show the results of test GG07 in which the swipe event again involved pure horizontal displacement. It is clear from Figure 6(b) that the horizontal load first increases at almost constant \(V\) until the parabolic surface is reached. The load path then follows the same parabola as that established from test GG03.

Figures 7(a) and 7(b) show an equivalent pair of tests, in which the swipe involved pure rotation. Test GG04 was after vertical loading to 1600N, and test GG08 after loading to 1600N and unloading to 200N. The pattern of behaviour, in terms of \(M / 2R\) and \(2R\theta\), is remarkably similar to that for the horizontal swipe tests in terms of \(H\) and \(u\). The main difference is that, in test GG08, the track along the parabola involves an increasing vertical load, whilst in GG07 the vertical load decreased. The yield surface in this plot is approximated by:

\[
\frac{M}{M_o} = 4 \frac{V}{V_o} \left( 1 - \frac{V}{V_o} \right)
\]  

...(4)

where \(M_o\) is the maximum moment, and \(M_o / 2R = 0.11V_o\). Again this value is close to the results of \(0.1V_o\) (Butterfield, 1980) and \(0.09V_o\) (Gottardi and Butterfield, 1993) for rectangular footings.

In four tests (see Table 1), similar swipe events were performed at a series of different vertical load levels with the aim of investigating possible variations of the shape of yield surface with hardening. It would obviously be convenient if the yield surface were to change size but not shape with penetration, as this would simply allow an envelope normalised with respect to the maximum past vertical load to be used. In Figure 8 the available results for horizontal swipe tests are compared: in test GG09 three independent swipe events were performed at \(V = 800N, 1600N\) and \(2000N\), and in test GG24 three similar events at \(V = 600N, 1200N\) and \(1800N\).
After each event the footing was moved back to the initial horizontal position and the penetration resumed. For the first two events of each test the footing was moved sideways less than 1.5mm, whilst in the final stage it was moved further (6mm). Test GG03, already shown in Figure 5 is also included. Although there is some evidence for a change in shape of the yield surface at different \( V_o \) values, the overall effect is that the surface expands with essentially the same parabolic shape.

There are two reasons why the load paths in the swipe tests do not give the shape of the yield surface exactly. Firstly, the elastic stiffness of the soil is not infinitely large, and secondly the testing rig is not infinitely stiff, so that when the vertical load reduces a small additional penetration occurs. Experimental data as measured in swipe tests, if it is to be used to represent a yield surface in the load space, should therefore be corrected to take into account the above effects.

The load path in the swipe test would follow the yield surface corresponding to the initial \( V_o \) value only if \( \delta w_p \) were infinitesimal and monotonically increasing throughout the test (some infinitesimal increase of \( w_p \) is necessary so that the load point is known to be on, rather than within, the yield surface). In practice:

\[
\delta w_p = \delta w - \delta w_e
\]  

...(5)

where \( \delta w \) is the actual measured vertical displacement of the footing, which is not quite zero due to the finite rig stiffness and the reduction in vertical load, and \( \delta w_e \) is the elastic part of the displacement, which can be estimated from the vertical elastic stiffness and the reduction in \( V \). Using an estimate of \( k_e \), one can therefore compute the additional plastic penetration \( \delta w_p \) for a given change in vertical load and measured \( \delta w \). If the form of the hardening law for \( V_o \) against \( w_p \) is known, then a correction can be applied to update \( V_o \) from its value at the beginning of the test. If equation (2) is used for the hardening law this correction can be carried out analytically. Figure 9 shows the results for test GG03, normalised by division by \( V_o \), with and without this correction. The rig flexibility accounts for some 10% change of \( V_o \), whereas the effect of the soil elasticity is almost negligible. The data from all swipe tests have been corrected for both effects in the following analyses and are presented in terms of the dimensionless parameters \( v = V / V_o \), \( m = M / 2RV_o \) and \( h = H / V_o \).

The data from all swipe tests starting at \( V =1600N \), normalised in this way, are plotted on the \((m,h)\) plane in Figure 10. The cross-section of the yield surface appears to be approximately elliptical. The ellipse is centred on the origin and the principal axes are slightly rotated anticlockwise from the co-ordinate axes. Curves in this plot do not start exactly from the origin because small transverse loading components were present at the end of the first vertical loading phase. Only the six main swipe tests are shown in the figure. Each test could also be plotted with the signs of both \( h \) and \( m \) reversed, so completing the other half of the ellipse (test GG04 is shown in the figure with the signs reversed).

Assuming the elliptical cross section, the generalisation of equations 3 and 4 to cases involving \((V,M / 2R,H)\) loading would be:
\[
\frac{m^2}{m_0^2} + \frac{h^2}{h_0^2} - 2a \frac{mh}{m_0 h_0} - (4v(1-v))^2 = 0
\]  
\[\ldots(6)\]

and the values of \(m_0\), \(h_0\) and \(a\) can be found by carrying out a least-squares fit of all the data points on the yield surface to equation 6. The resulting values are \(m_0 = 0.090\), \(h_0 = 0.1213\) and \(a = -0.2225\). The equivalent values of these parameters established from load-controlled tests for a rectangular footing (100mm \(\times\) 500mm) on dense sand were 0.088, 0.130 and -0.22 respectively (Butterfield and Gottardi, 1994).

Defining \(m_1 = m/(4v(1-v))\) and \(h_1 = h/(4v(1-v))\), the complete three-dimensional yield surface can be represented by the ellipse:

\[
\frac{m_1^2}{m_0^2} + \frac{h_1^2}{h_0^2} - 2a \frac{h_1 m_1}{h_0 m_0} - 1 = 0
\]  
\[\ldots(7)\]

The entire data for the swipe tests (including each test with signs reversed) are presented in this way in Figure 11, where it can be seen that the experimental data fall close to the best fit ellipse. Data for \(v > 0.9\) and \(v < 0.1\) are omitted from this figure, since in these cases the division by a small factor exaggerates minor errors. The tests starting at \(V = 200N\) start within the elastic region and move out to the yield surface, whereas those starting at \(V \approx 1600N\) lie entirely on the surface.

The quality of fit of the parabolic section can be studied by defining the quantity:

\[
q = \sqrt{\frac{m^2}{m_0^2} + \frac{h^2}{h_0^2} - 2a \frac{hm}{h_0 m_0}}
\]  
\[\ldots(8)\]

so that the yield surface becomes the parabola \(q - 4v(1-v) = 0\) (Butterfield and Gottardi, 1996). The data for the swipe tests are plotted in the \((v, q)\) plane in Figure 12, which shows that the parabolic section is indeed a close fit to it. Again the tests starting at \(V = 200N\) start within the yield surface.

Figures 11 and 12 confirm that equation 6 is a good approximation to the shape of the yield surface. The expression is of the same form as observed by Gottardi and Butterfield (1993) from load-controlled tests on rectangular footings on dense sand. It is also very similar to that used by Martin (1994) for combined loading tests of model spudcans on soft clays. It is remarkable that a similarly shaped envelope applies to foundations of differing geometries on such very different soils. Not only is the algebraic form similar, but the parameter values also imply comparable aspect ratios of the yield surface for these rather dissimilar systems.

Comparison of the observed ratios of displacements with the ratios of loads allows the shape of the plastic potential in the \((M/2R, H)\) plane to be investigated. Discussion of the flow rule is given at the end of Section 4.4.
4.3. Radial Displacement Tests

Radial Displacement Tests involve application of a straight displacement path in \((w,2R\theta,u)\) space. Clearly one special case is that of pure vertical penetration, which has already been discussed. These tests complement swipe tests, in that they give information about the hardening rule, and about the flow rule. For the latter they allow all three components \((dw,2Rd\theta,du)\) to be determined. The drawback is that this information is established only at predetermined points in the displacement vector space, so that many tests are required to build up a complete picture.

Six monotonic radial displacement tests were carried out, with different inclination either on the \((w,2R\theta)\) plane or on the \((w,u)\) plane, see Table 1. Figure 13(a) shows the load-displacement plot for the three tests involving combined horizontal and vertical movement, and Figure 13(b) shows the load paths for the same tests (together with one of the vertical penetration tests). These are approximately straight until they approach failure. As the ratio \(du/dw\) increases, so does the slope of the load path \(dH/dV\), as would be expected.

Figures 14(a) and 14(b) show the equivalent data for three tests involving combined rotation and vertical movement. The pattern of behaviour is very similar, except that the load paths show more curvature.

The question arises as to whether the load-deflection behaviour observed in the radial displacement tests can be understood in the context of the vertical load-deflection response as represented by equation (2). If the expansion of the yield surface is only a function of the plastic vertical penetration then equation (2) can be applied to more general loading cases simply by replacing \(V\) by \(V_0\). For each combination of \((V,M/2R,H)\) values, equation 6 can be used to derive the following expression for the current value of \(V_0\):

\[
V_0 = \frac{4V^2}{4V - \sqrt{\frac{m^2}{m_0^2} + \frac{H^2}{h_0^2} - \frac{aMH}{Rm_0h_0}}}
\] ...

(9)

Figure 14 shows the value of \(V_0\) (calculated using equation (9) and best-fit parameters derived from the swipe tests) for the three radial loading tests involving horizontal displacement, plotted against plastic vertical penetration \(w_p = w - V/k\). The experimental data are compared with equation 2, derived from the vertical loading tests. In order to eliminate bedding errors at small vertical loads, the curves are matched at \(V_0 = 1000\)N. It is apparent from the figure that, whilst the experimental curves are broadly similar in shape to the analytical expression, they differ in detail. This implies that the assumption that the hardening of the yield surface is solely a function of \(w_p\) is an oversimplification, but could nevertheless be used as an initial approximation to the data.

Figure 16 shows a similar plot for the three radial displacement tests involving moment loading. The quality of the fit of the experimental curve to the data is rather better, indicating that for these cases the simple assumption about hardening of the yield surface is more justifiable.
Radial displacement tests also provide limited information about the flow rule (see Section 4.4), but each test generates data only at one ratio of displacements. The curvature of the load paths is related to changes in shape of the plastic potential as the load increases. Gottardi and Butterfield (1995) performed radial load path tests, and observed a curvature of the displacement path in the opposite sense to that of the load paths reported here. These two observations are, of course, entirely consistent.

4.4. Constant Vertical Load Tests

Constant V tests involve an initial phase of purely vertical displacement as for the swipe tests. The only difference is that during the horizontal movement or rotation phase, instead of maintaining a fixed vertical displacement, the vertical load is held constant. Such tests are analogous to constant $p^*$ triaxial tests on clays. Constant V tests can be carried out with or without a reduction in the vertical load below its maximum value before the horizontal movement or rotation stage. The test programme included only four such tests, all at $V = 1600$N, see Table 1.

Because the apparatus used is displacement controlled, a constant $V$ test has to be achieved by feedback control. This was done by continuously monitoring the vertical load, and adjusting the vertical velocity to maintain $V$ as close as possible to the prescribed value. This procedure was able to maintain $V$ to within acceptable limits ($\pm 10$ N).

Figure 17(a) and 17(b) show the load-displacement curve and the displacement path for test GG14. As the footing is displaced horizontally it also penetrates vertically. This result is entirely consistent with the fact that the swipe test (at constant vertical displacement) involves a reduction of $V$. During the test the horizontal load increases and then decreases, returning to a negligible value by the end of the test. An interesting question is whether this rather intriguing behaviour can be understood within the framework already established. As for the radial loading tests, a continually changing value of $V_0$, and also of the vertical plastic penetration, can be calculated as the test proceeds. Figure 18 shows the resulting plot of $V_0$ against $w_p$, in which the results for the four constant $V$ tests are compared with the hardening law derived from the vertical penetration tests. The curves are matched at the start of the constant $V$ phase. It can be seen that the hardening law given by equation (2) matches the results of the tests quite well. In particular, each of the experimental curves shows a peak rather similar to that given by equation (2). Again the detail differs slightly, indicating that a more complex hardening law is needed to explain all aspects of the tests.

The fact that the post-peak value of $V_0$ has dropped to 1600N (the constant $V$ value) explains why $H$ falls to zero in Figure 16(a). Towards the end of the test, the feedback system was increasingly unable to maintain a constant $V$ value. Once $V_0$ falls below 1600N then it is expected that it would be impossible to sustain $V$ as a constant during subsequent horizontal displacement.

Constant $V$ tests also provide information about the flow rule in the $(V, M/2K)$ or $(V, H)$ plane, in that the observed plastic displacement rate ratios can be compared to the force ratios.
For instance, Figure 17(b) shows that, up to the peak $H$ value, the displacement path is approximately straight, implying that the displacement rate ratio is, perhaps surprisingly, insensitive to the force ratio for this load path. This is in agreement with the observations of Gottardi and Butterfield (1995).

The constant $V$ and radial displacement tests presented here do not provide conclusive evidence about the flow rule, but it is clear that an associated flow rule on these planes is not supported by the data. The same observation was made by Martin (1994) and Gottardi and Butterfield (1995).

The possibility of an associated flow rule in the $(M/2R,H)$ plane may be tested by calculating the ratios $2RH/M$ and $du_p/2Rd\theta_p$. If the yield surface is expressed as $f(V,M/2R,H) = 0$, then standard plasticity theory gives that, for an associated flow rule, $2Rd\theta_p = \Lambda d\phi / d(M/2R)$ and $du_p = \Lambda d\phi / dH$ where $\Lambda$ is a multiplier determined by the hardening law. Using equation (6), the plastic strain rate ratio can therefore be calculated as:

$$\frac{du_p}{2Rd\theta_p} = \frac{2RH_m - Mah_0}{Mh_0 - 2RH_m}$$

Figure 19 shows a comparison of the data with the predicted relationship between the displacement rate ratio and the force ratio, equation (10). Experimental data is presented from the six swipe tests at $V = 1600N$, and the four constant $V$ tests. To avoid the need to subtract the elastic displacement rates from the total rates to obtain the plastic displacement rates, the data are just plotted for one measurement of the relevant slope near the peak in $H$ and $M/2R$. At this point the rates of change of $H$ and $M/2R$ are zero, so the rates of change of $u_e$ and $2R\theta_e$ are also zero, and the plastic displacement rate is equal to the total displacement rate. A very minor error is introduced because the peaks in $H$ and $M/2R$ occur at slightly different points during the test. Figure 19 demonstrates that for all but one of the 10 tests presented (the exception being GG15), the observed force ratio corresponds closely to the expected value using the applied displacement rate ratio and an associated flow rule. An associated flow rule in the $(M/2R,H)$ therefore appears to be justified.

5. CONCLUSIONS

The results from a series of displacement-controlled tests on model footings on dense sand have been presented. The tests were designed to investigate footing behaviour under combined loading, and to provide information to allow plasticity models for the problem to be devised. The principal conclusions drawn from the tests are:

1. The tests can be successfully interpreted in the context of hardening plasticity theory in terms of force resultants and overall footing displacements.
2. The shape of the yield surface in $(V,M/2R,H)$ space is well described by a parabolic ellipsoid, as observed previously by Gottardi and Butterfield (1993) and Martin (1994) for other foundation and soil types.
3. The hardening behaviour can be defined by expressing the size of the yield surface as a function of plastic vertical penetration. The necessary expression can be derived from a
vertical loading test. This simplification can provide a qualitative explanation of both radial loading tests and constant V tests, but detailed quantitative modelling would require a more complex relationship.

4. An associated flow rule in the \((M/2R, H)\) plane is supported by the tests, but quantitative conclusions about the flow rule in the \((V, M/2R)\) and \((V, H)\) planes are not drawn, although it is clearly not associated on these planes.

5. The fact that the above interpretation, in particular the full description of a satisfactory yield surface, is supported by a large quantity of high quality data obtained from tests on a variety of footings and soil types, suggests that, when fully developed, this approach will provide a highly satisfactory framework for understanding one of the fundamental problems in geotechnical engineering.

6. ACKNOWLEDGEMENT

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7. REFERENCES


<table>
<thead>
<tr>
<th>Test Group</th>
<th>Sub-Group</th>
<th>Tests</th>
<th>Notes</th>
<th>tan⁻¹(\frac{2Rd\theta}{du}) (degrees)</th>
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<tr>
<td>Vertical</td>
<td>Beyond peak, with unload-reload cycles</td>
<td>GG01, (D_r = 79%), GG02, GG11, GG26</td>
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<td>Loading</td>
<td>Up to (V=1600)N, with unload-reload at (V=1000)N</td>
<td>GG03, GG04, GG05, GG06, GG07, GG08, GG10, GG12, GG14, GG15, GG18, GG19, GG25, GG28, GG29</td>
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<td>GG05</td>
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Table 1: Details of Tests
Figure 1: Sign conventions for loads and displacements

[Photograph of the rig]

Figure 2: General view of testing apparatus
Figure 3: Load paths for different classes of tests

Figure 4: Vertical loading tests
Figure 5: Schematic diagram of load paths followed in swipe tests

Figure 6: Results of swipe tests with horizontal displacement
Figure 7: Results of swipe tests with rotation

Figure 8: Horizontal swipe tests at different load levels
Figure 9: Corrections to swipe tests to give shape of yield surface

Figure 10: Load paths for swipe tests from $V = 1600N$
Figure 11: Elliptical section of yield surface
Figure 12: Parabolic section of yield surface

Figure 13: Radial displacement tests (horizontal loading)
Figure 14: Radial displacement tests (moment loading)

Figure 15: Comparison of radial displacement (horizontal) tests with hardening law
Figure 16: Comparison of radial displacement (rotation) tests with hardening law
Figure 17: Constant vertical loading test with horizontal displacement
Figure 18: Comparison of constant V tests with hardening law

Figure 19: Comparison between observed and predicted force ratios