Derivation of Incremental Stress-Strain Response for Plasticity Models Based on Thermodynamic Functions

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1. Introduction

The constitutive behaviour of soils and other granular materials is often usefully described within the framework of plasticity theory. The existence of a yield surface in stress space, enclosing a region in which the strains are substantially elastic, is well established empirically. The validity of the concepts of a flow rule and hardening law are also well established for many granular materials. Within this broad framework, many constitutive models, of varying complexity, have been proposed. Recent research on the non-linearity of soils at small strains has raised important questions about the range of applicability of these models, but the basic framework remains unchallenged.

One of the most important steps in the development of plasticity theory was the development of a standard approach which allows the incremental stress-strain response to be derived, given the functional forms of (a) the elastic response, (b) the yield surface, (c) the plastic potential and (d) a hardening law. A model must be formulated incrementally if it is to be used in non-linear finite element analysis, which is undoubtedly the most important technique currently available for the analysis of engineering problems involving materials with complex constitutive behaviour. For a standard derivation of incremental response for conventional plasticity models see for example Zienkiewicz (1977).

A feature of most plasticity models for granular materials is that, unlike models for metals, they employ "non-associated" flow rules, that is the plastic potential is not the same as the yield surface. This is essential if the dilative properties of granular materials are to be modelled realistically. Unfortunately it is possible to devise non-associated plasticity models which violate thermodynamic principles, and no simple procedure has been devised to apply retrospective checks to plasticity models to check that they are thermodynamically acceptable.

The above problem has led to an alternative approach to plasticity modelling of granular materials (Houlsby 1981, 1982). In this approach the entire constitutive behaviour is determined from two thermodynamic functions: the Helmholtz Free Energy and a Dissipation Function. By adopting the orthogonality principle (Ziegler, 1977),

which may be regarded as a more restrictive form of the Second Law of Thermodynamics, the entire constitutive behaviour may then be determined.

An important observation is that, for certain forms of the thermodynamic functions, the phenomena of elastic behaviour, a yield surface, flow rule and hardening law all appear as consequences of the above formulation and do not need to be imposed as separate features. The models derived are, however, guaranteed not to violate thermodynamic principles. An obstacle to use of this approach has been that, although incremental stress-strain response has been determined for specific models, no general technique has been established to derive the incremental response directly from the thermodynamic functions. The purpose of the paper is to set out a general technique (subject to one minor restriction on the form of the dissipation function), so that incremental response may be derived. This paper clears the way for the use of the new method, guaranteeing thermodynamic acceptability, in non-linear finite element analysis.

2. Functions of Kinematic Variables

The most important feature of the approach to constitutive modelling adopted here is that the entire constitutive behaviour is determined by the specification of only two functions: the Helmholtz Free Energy and the Dissipation Function. Both of these are functions only of kinematic variables, i.e. the strains and internal kinematic variables such as the plastic strain. Once the form of the functions is determined then the entire constitutive behaviour may be derived purely by manipulation of these functions, and no further assumptions need to be made.

Let the strains be denoted by \( \varepsilon_i \) and the internal variables by \( \alpha_j \). For convenience a single subscript notation is used here, with stresses and strains treated as vectors, as is usual in development of finite element code. In general the Helmholtz Free Energy may be a function of both the strains and the internal variables \( F = F(\varepsilon_i, \alpha_j) \). The Dissipation Function may not only be a function of these variables, but also their rates, thus \( D = D(\varepsilon_i, \alpha_j, \dot{\varepsilon}_i, \dot{\alpha}_j) \). If the material is rate-independent, then it can easily be shown that the Dissipation Function must be homogeneous and of first order in the rates, so that \( D(\varepsilon_i, \alpha_j, n\dot{\varepsilon}_i, n\dot{\alpha}_j) = nD(\varepsilon_i, \alpha_j, \dot{\varepsilon}_i, \dot{\alpha}_j) \).

Applying the standard procedures developed for this approach (Ziegler (1977), Houlsby (1981, 1982)), the constitutive behaviour is then entirely determined by the pair of equations:

\[
\sigma_i = \frac{\partial F}{\partial \varepsilon_i} + \frac{\partial D}{\partial \dot{\varepsilon}_i}
\]  
\[
0 = \frac{\partial F}{\partial \alpha_i} + \frac{\partial D}{\partial \dot{\alpha}_i}
\]  

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Unfortunately, however, these equations are not immediately in an appropriate form for expressing the constitutive behaviour for application in numerical computation, and considerable manipulation (see Houlsby (1981, 1982)) is required to achieve this.

Whilst it has always proved possible to derive the constitutive behaviour from the above expressions, this has hitherto been by application of ad hoc procedures, and no standard method has been developed. A particularly important form of the constitutive equations is an incremental relationship between the stress and strain. This form is used, for example, in non-linear finite element analysis to obtain the stiffness matrix. The main purpose of this paper is to develop a standard analysis procedure (for slightly restricted cases) by which the incremental response may be derived.

Firstly it is useful to re-cast the forms of the Helmholtz Free Energy and Dissipation Function. It is often convenient to consider internal variables which correspond to the conventional plastic strains \( \alpha_i = \varepsilon_i^p \). The elastic strain is then defined through the standard decomposition of strain into elastic and plastic components \( \varepsilon_i = \varepsilon_i^e + \varepsilon_i^p \). It is then convenient to redefine the Helmholtz Free Energy as a function of elastic and plastic strains \( F = F(\varepsilon_i^e, \varepsilon_i^p) \), since this often gives a simple functional form. This change is purely for convenience, and does not restrict in any way the form of \( F \).

The Dissipation Function is also re-written in terms of elastic and plastic strains. In addition the restriction is made that the Dissipation Function is not a function of the elastic strain rates. This simplification is not unduly restrictive, but it eliminates the possibility of materials which exhibit changes of stress with no change of strain (i.e. rigid responses). In most developments of constitutive models based on thermodynamics this restriction is assumed without question or discussion. The form of the Dissipation Function therefore becomes \( D = D(\varepsilon_i^e, \varepsilon_i^p, \varepsilon_i^p) \).

The standard approach for defining the constitutive behaviour needs to be modified slightly to account for the introduction of elastic and plastic strains. A simple way is to introduce a set of constraint functions \( C_i = \dot{\varepsilon}_i - \dot{\varepsilon}_i^e - \dot{\varepsilon}_i^p = 0 \), and to modify the formulation to:

\[
\sigma_i = \frac{\partial F}{\partial \varepsilon_i} + \lambda_j \frac{\partial C_j}{\partial \varepsilon_i} + \frac{\partial D}{\partial \varepsilon_i^e} + \frac{\partial D}{\partial \varepsilon_i^p}
\]

\[
0 = \frac{\partial F}{\partial \varepsilon_i^e} + \lambda_j \frac{\partial C_j}{\partial \varepsilon_i^e} + \frac{\partial D}{\partial \varepsilon_i^e} + \lambda_j \frac{\partial D}{\partial \varepsilon_i^p}
\]

\[
0 = \frac{\partial F}{\partial \varepsilon_i^p} + \lambda_j \frac{\partial C_j}{\partial \varepsilon_i^p} + \frac{\partial D}{\partial \varepsilon_i^p} + \lambda_j \frac{\partial D}{\partial \varepsilon_i^p}
\]
where $\lambda_i$ is a Lagrangean multiplier. Note that throughout this paper the summation convention is used for a repeated index. From equation (3) it rapidly follows that $\lambda_i = \sigma_i$, so that (4) and (5) can be rewritten:

$$\sigma_i = \frac{\partial F}{\partial \varepsilon_i}$$  \hspace{1cm} (6)

$$\sigma_i = \frac{\partial F}{\partial \varepsilon_i} + \frac{\partial D}{\partial \varepsilon_i}$$  \hspace{1cm} (7)

In this paper the constitutive behaviour will be derived entirely from the thermodynamic functions in their original form. An alternative approach is to use the powerful technique of Legendre Transformations to change from the original functions of kinematic variables to new functions of stress variables. This alternative approach (Collins and Houlshby (1996), Collins (1996)) proves to be valuable in that it provides useful insights into the relationships between the forms of the Dissipation Function, the yield surface and the plastic potential.

3. Derivation of Incremental Response

In the following development it will be useful to use the shorthand $F_{ij} = \frac{\partial F}{\partial \varepsilon_i}$.

$$F_{ij}^{ep} = \frac{\partial^2 F}{\partial \varepsilon_i \partial \varepsilon_j^{ep}} \text{ etc.}$$  \hspace{1cm} (8)

Note then that the differential of (6) gives:

$$\sigma_i = F_{ij}^{ep} \dot{\varepsilon}_j + F_{ij}^{ep} \dot{\varepsilon}_i^p = F_{ij}^{ep} \dot{\varepsilon}_j + \left(F_{ij}^{ep} - F_{ij}^{ep}\right) \dot{\varepsilon}_j^p$$

Consider now the form of $D$. Although $D$ is first order in the plastic strain rates, it is not usually linear, but $D^2$ is not only second order, but for many models of material behaviour it is quadratic (see Section 4 below for examples). This restriction will be applied in the following development. For this case it is useful to define:

$$q_{ij} = \frac{1}{2} \frac{\partial^2 (D^2)}{\partial \varepsilon_i \partial \varepsilon_j}$$  \hspace{1cm} (9)

where $q_{ij} = q_{ij} (\varepsilon_i, \varepsilon_i^p)$, i.e. $q_{ij}$ is not a function of the plastic strain rates. Define also the inverse of $q_{ij}$ as $p_{ij}$:

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\[ p_{ij} = \delta_{ik} \]  

In some cases it will be convenient to determine the inverse analytically, in other cases it can be determined numerically. Note that \( q_{ij} \) and \( p_{ij} \) are of course symmetric. Clearly the existence of the inverse depends on the assumption that the determinant of \( q_{ij} \) is non-zero. In fact certain important forms of \( D \) result in a zero determinant, with these being cases where there is a mode of plastic deformation which would induce no dissipation. This deformation mode is then suppressed by a side constraint that it must be zero. Models involving either zero or constant plastic dilation come into this category. One approach to their treatment is to reduce the number of internal variables by one. In this case the internal variables no longer correspond precisely to the conventional plastic strains (although they are closely related to them) and the modified formulation described in Appendix A is appropriate.

Multiplying (7) by \( \dot{\varepsilon}_i^p \) gives:

\[ \left( \sigma_i - F_i^p \right) \dot{\varepsilon}_i^p = \frac{\partial D}{\partial \dot{\varepsilon}_i^p} \dot{\varepsilon}_i^p = D \]  

where the second part of equation (11) follows from the fact that \( D \) is first order in the strain rates. Note now that because \( \frac{\partial D^2}{\partial \dot{\varepsilon}_i^p} \) is also first order in the rates it follows that:

\[ q_{ij} \varepsilon_j^p = \frac{1}{2} \frac{\partial^2 D^2}{\partial \dot{\varepsilon}_i^p \partial \dot{\varepsilon}_j^p} \dot{\varepsilon}_i^p = \frac{1}{2} \frac{\partial D^2}{\partial \dot{\varepsilon}_i^p} = D \frac{\partial D}{\partial \dot{\varepsilon}_i^p} \]  

Multiplying (7) by \( D \) therefore gives:

\[ \left( \sigma_i - F_i^p \right) D = \frac{\partial D}{\partial \dot{\varepsilon}_i^p} = q_{ij} \dot{\varepsilon}_j^p \]  

or, defining \( r_j = p_{ij} \left( \sigma_i - F_i^p \right) \):

\[ p_{ij} \left( \sigma_i - F_i^p \right) D = r_j D = \dot{\varepsilon}_j^p \]  

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Equations (8), (11) and (14) are $2n+1$ equations in the $3n+1$ variables $\sigma_i$, $\dot{\varepsilon}_i$, $\dot{\varepsilon}_i^p$ and $D$ (where $n$ is the number of strain variables), and at first sight $\dot{\varepsilon}_i^p$ and $D$ could be eliminated to give $n$ equations representing the stiffness relationship between $\sigma_i$ and $\dot{\varepsilon}_i$. However, (11) and (14) are not linearly independent equations in the rates, and they can be combined to give either $D = 0$ (which is the case of no dissipation, when all plastic strains are zero and elastic behaviour occurs) or:

$$p_{ij}(\sigma_i - F_i^p)(\sigma_j - F_j^p) = 1$$  \hspace{1cm} (15)

which is simply the yield condition. Thus it is seen that the existence of the yield surface arises as a natural consequence of the formulation.

To obtain $2n+1$ useful equations in the rates it is necessary to drop equation (11) and introduce instead the consistency condition that the stress point remains on the yield surface during plastic deformation. This is given by the differential of (15):

$$\dot{p}_{ij}(\sigma_i - F_i^p)(\sigma_j - F_j^p) + 2p_{ij}(\sigma_i - F_i^p)(\dot{\sigma}_j - F_{jk}^p\dot{\varepsilon}_k^p - F_{jk}^p\dot{\varepsilon}_k^p) = 0$$  \hspace{1cm} (16)

Noting that $\dot{p}_{ij} = -p_{ik}p_{jk}\dot{q}_{kl}$, this can be re-arranged to:

$$r_k\dot{\varepsilon}_k = 2r_j\left(\sigma_j - F_{jk}^p\dot{\varepsilon}_k - \left(F_{jk}^{pp} - F_{jk}^{pe}\right)\dot{\varepsilon}_k^p\right)$$  \hspace{1cm} (17)

Defining $q_{jk}^{pe} = \frac{\partial q_{ij}}{\partial \varepsilon_k}$ etc., and substituting for $\dot{\varepsilon}_i^p$, (17) can further be re-written as:

$$r_k\dot{\varepsilon}_k \left( q_{klm}^p\dot{\varepsilon}_m + \left(q_{klm}^p - q_{klm}^{pe}\right)r_mD \right) = 2r_j\left(\sigma_j - F_{jk}^p\dot{\varepsilon}_k - \left(F_{jk}^{pp} - F_{jk}^{pe}\right)\dot{\varepsilon}_k^p\right)$$  \hspace{1cm} (18)

Equation (8) may be rewritten in the form:

$$\dot{\sigma}_i = E_{ij}^{ce}\dot{\varepsilon}_j + r_j\left(E_{ij}^{cp} - F_{ij}^{ce}\right)D$$  \hspace{1cm} (19)

which can be substituted into (18) to give, on rearrangement:

$$r_k\dot{\varepsilon}_k \left( r_m\left(q_{klm}^p - q_{klm}^{pe}\right) - 2\left(F_{kl}^{cp} - F_{kl}^{ce} - F_{kl}^{pp} + F_{kl}^{ce}\right)\right)D =$$

$$r_i\left(-r_j\dot{q}_{ij}^{pe} + 2\left(F_{ij}^{ce} - F_{ij}^{pe}\right)\dot{\varepsilon}_j\right)$$  \hspace{1cm} (20)
This is in effect an equation of the form $AD = B \dot{\varepsilon}$, which can be used to substitute for $D$ into (19), giving after some minor manipulation:

$$\sigma_i = \left( F_{ij}^{ee} + r_k \left( F_{ik}^{ep} - F_{ik}^{ee} \right) \frac{B_j}{A} \right) \dot{\varepsilon}_j = d_{ij} \dot{\varepsilon}_j$$  \hspace{1cm} (21)

where $d_{ij}$ is the required stiffness matrix. All the terms are easily determined from the original functions. Furthermore, equations (20) and (21) have been derived for very general forms of the thermodynamic functions, and in many simple cases a number of the terms are zero.

### 4. Examples

Three examples will be pursued here: (a) a very simple analogue of elastic-perfectly plastic (cohesive) behaviour, (b) a critical state model appropriate for soft clays, and (c) a general model for frictional behaviour which is appropriate for a loose granular material. The first two cases involve associated flow, the third case non-associated.

#### 4.1 COHESIVE ELASTIC PERFECTLY-PLASTIC MODEL

Consider first a simple 2-dimensional analogue of plasticity in which the strain vector is $\gamma_i = [\gamma_1 \gamma_2]^T$ and stress vector $\tau_i = [\tau_1 \tau_2]^T$. The use of $\gamma$ and $\tau$ in place of $\varepsilon$ and $\sigma$ is simply to follow convention for shear components. The plastic behaviour is given by $\tau_i = G\gamma_i^p$, and the yield surface (and plastic potential) is $\tau_i \gamma_i = c^2$. The thermodynamic functions required to derive this model are:

$$F = \frac{G}{2} \gamma_i^p \gamma_i^p$$  \hspace{1cm} (21)

$$D = c \sqrt{\gamma_i^p \gamma_i^p}$$  \hspace{1cm} (22)

From these expressions the standard manipulation rapidly gives $F_i^e = G\gamma_i^p$, $F_i^p = 0$. $F_{ij}^{ee} = G \delta_{ij}$, $F_{ij}^{ep} = 0$, $F_{ij}^{ee} = 0$, $F_{ij}^{pp} = 0$, $q_{ij} = c \delta_{ij}$, $p_{ij} = \frac{1}{3} \delta_{ij}$, $r_i = \frac{\sigma_i e}{c}$, $q_{ijk} = 0$ and $q_{ijk}^p = 0$. Equation (8) gives the required elastic behaviour and equation (15) gives the required form of the yield locus. The factors in equation (21) are determined as $A = 2G \frac{\tau_i \gamma_j}{c^2} = 2G$ and $B_j = 2G \frac{\tau_i}{c} \tau_j$, so that $d_{ij} = G \left( \delta_{ij} - \frac{\tau_i \tau_j}{c^2} \right)$, which can easily be verified as the correct plastic stiffness matrix.
4.2 CRITICAL STATE MODEL

The second example is that of the “Modified Cam-Clay” critical state model which is often used to model the work-hardening behaviour of normally consolidated and lightly overconsolidated clays. Houltsby (1981) demonstrates how this model can be derived from the thermodynamic functions. For triaxial stress states it is convenient to express the model in terms of the stress and strain vectors $[p \ q \ v]^T$ and $[\dot{v} \ \dot{\varepsilon} \ \varepsilon]^T$ commonly used in critical state soil mechanics. Using a slightly simplified form of the functions used by Houltsby (1981), with the simplification being related to the datum chosen for plastic strain, the required functions are:

$$F = p_r \kappa \exp\left(\frac{v_p}{\kappa}\right) + \frac{3G \varepsilon_r^2}{2} + p_r (\lambda - \kappa) \exp\left(\frac{v_p}{\lambda - \kappa}\right)$$

$$D = p_r \exp\left(\frac{v_p}{\lambda - \kappa}\right) \left(\dot{\varepsilon}_p^2 + M^2 \dot{\varepsilon}_p^2\right)$$

where $p_r$ is a reference pressure, $M$, $\lambda$ and $\kappa$ are material parameters commonly used in critical state soil mechanics (except that $\lambda$ and $\kappa$ refer to slopes in $(\ln p, \ln v$) space rather than in $(\ln p, V$) space) and $G$ is the shear modulus. Note that the model defines a bulk modulus proportional to pressure, but a constant shear modulus. It follows that $F^c = [p \ q]^T = \left[p_r \exp\left(\frac{v_p}{\kappa}\right) \ 3G \varepsilon_r^T\right]^T$, $F^p = \left[p_r \exp\left(\frac{v_p}{\lambda - \kappa}\right) \ 0\right]^T$. It is convenient to introduce the shorthand $p_x = p_r \exp\left(\frac{v_p}{\lambda - \kappa}\right)$. It then follows that

$$F_{ij}^c = \begin{bmatrix} \frac{p}{\kappa} & 0 \\ 0 & 3G \end{bmatrix}, \quad F_{ij}^p = 0, \quad F_{ij}^{pp} = \begin{bmatrix} \frac{p_x}{\lambda - \kappa} & 0 \\ 0 & 0 \end{bmatrix}.$$ Differentiation of $D^2$ gives

$$q_{ij} = p_x^2 \begin{bmatrix} 1 & 0 \\ 0 & M^{-2} \end{bmatrix} \quad \text{and} \quad p_{ij} = p_x^2 \begin{bmatrix} 1 & 0 \\ 0 & M^{-2} \end{bmatrix},$$

so that equation 15 gives the yield locus as the ellipse \( \frac{(p - p_x)^2}{p_x^2} + \frac{q^2}{M^2 p_x^2} = 1 \). One can further derive the relations

$$r_i = p_x^2 \begin{bmatrix} q \\ M^2 \end{bmatrix}^T, \quad q_{ijk} = 0, \quad q_{ij1} = 2p_x^2 \begin{bmatrix} 1 & 0 \\ 0 & M^2 \end{bmatrix} \quad \text{and} \quad q_{ij2} = 0.$$ Noting that

$$r_k q_{ijk} = p_x^2 \begin{bmatrix} 1 & 0 \\ 0 & M^2 \end{bmatrix},$$

the factors in equation (20) are determined as

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\[ A = \frac{2}{p_x^3} \left( (p - p_x)^2 - \frac{p\lambda}{\kappa(\lambda - \kappa)} + \frac{q^2}{M^4} \left( \frac{M^2(p - p_x)}{(\lambda - \kappa)} + 3G \right) \right) \]

and

\[ B_j = \frac{2}{p_x^2} \left[ \frac{(p - p_x)^2}{\kappa} \frac{3Gq}{M^2} \right]^T \]

Finally, the stiffness matrix can be determined as:

\[
d_{ij} = \begin{bmatrix} \frac{p}{\kappa} & 0 \\ 0 & 3G \end{bmatrix} - \frac{2}{Ap_x^2} \begin{bmatrix} \frac{p^2}{\kappa} \frac{(p - p_x)^2}{3Gq(p - p_x)} \frac{3Gq(p - p_x)}{M^2 \kappa} \\ \frac{3Gq(p - p_x)}{M^2 \kappa} \frac{9G^2q^2}{M^4} \end{bmatrix} \quad (25)\]

Equation (25) can be manipulated into a variety of alternative forms. It has been verified that equation (25) is correct by comparing it with values from a totally different approach used to derive a compliance matrix in Houlsby (1981).

4.3 A FRICTIONAL PLASTICITY MODEL

The last example is a model of a linear elastic, perfectly-plastic frictional model with zero dilation and a yield surface the same shape as the failure surface proposed by Matsuoka and Nakai (1974). In this case, if the plastic strains are used as the internal variables, the matrix \( q_{ij} \) is singular because there is no dissipation associated with the volumetric mode of plastic deformation: this mode is suppressed by the zero dilation constraint. Use of constraints to define dilation conditions is discussed by Houlsby (1992). It is convenient therefore to reduce the number of internal variables by one, and use the modified formulation described in Appendix A. The derivation is carried out for three principal stresses. The strain variables are therefore \( \varepsilon_i = [\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3]^T \), and the chosen internal variables are \( \alpha_i = [\alpha_1 \quad \alpha_2]^T = \left[ (\varepsilon_1 - \varepsilon_2) + (\varepsilon_2 - \varepsilon_3) + (\varepsilon_3 - \varepsilon_1) \right]^T \). The required thermodynamic expressions are:

\[
F = \frac{K}{2} v^2 + \frac{G}{3} \left( (\varepsilon_1 - \varepsilon_2 - \alpha_1)^2 + (\varepsilon_2 - \varepsilon_3 - \alpha_2)^2 + (\varepsilon_3 - \varepsilon_1 + \alpha_1 + \alpha_2)^2 \right) \quad (26)\]

\[
D = \mu \left[ \frac{1}{3} (\sigma_1\sigma_2\alpha_1^2 + \sigma_2\sigma_3\alpha_2^2 + \sigma_3\sigma_1(\alpha_1 + \alpha_2)^2) \right] \quad (27)\]

where \( v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \) and the alternative form in which the dissipation function is expressed in terms of stresses is used.
The intermediate results are given in Appendix B. Note, however, that although some of the manipulation is rather lengthy, all the expressions are cast in a form so that they can readily be evaluated by a symbolic manipulation programme. It follows from equations (B2) and (B6) that the yield locus is given by:

$$\sigma_1 (\sigma_2 - \sigma_3)^2 + \sigma_2 (\sigma_3 - \sigma_1)^2 + \sigma_3 (\sigma_1 - \sigma_2)^2 = 8\mu^2 \sigma_1 \sigma_2 \sigma_3$$  \hspace{1cm} (28)

The final form of the stiffness matrix is:

$$d_y = K \begin{bmatrix} C - D \sigma_2 \sigma_3 & C - D \sigma_2 \sigma_3 & C - D \sigma_2 \sigma_3 \\ C - D \sigma_3 \sigma_1 & C - D \sigma_3 \sigma_1 & C - D \sigma_3 \sigma_1 \\ C - D \sigma_1 \sigma_2 & C - D \sigma_1 \sigma_2 & C - D \sigma_1 \sigma_2 \end{bmatrix} + \frac{2G}{E} \begin{bmatrix} \sigma_1^2 (\sigma_2 - \sigma_3)(\sigma_2^2 - \sigma_3^2) & \sigma_2^2 (\sigma_3 - \sigma_1)(\sigma_2^2 - \sigma_3^2) & \sigma_3^2 (\sigma_1 - \sigma_2)(\sigma_2^2 - \sigma_3^2) \\ \sigma_2^2 (\sigma_1 - \sigma_3)(\sigma_1^2 - \sigma_3^2) & \sigma_3^2 (\sigma_3 - \sigma_1)(\sigma_1^2 - \sigma_3^2) & \sigma_1^2 (\sigma_2 - \sigma_3)(\sigma_1^2 - \sigma_3^2) \\ \sigma_3^2 (\sigma_1 - \sigma_3)(\sigma_1^2 - \sigma_3^2) & \sigma_1^2 (\sigma_2 - \sigma_3)(\sigma_1^2 - \sigma_3^2) & \sigma_2^2 (\sigma_1 - \sigma_2)(\sigma_1^2 - \sigma_3^2) \end{bmatrix}$$  \hspace{1cm} (29)

where the factors $C$, $D$ and $E$ are functions of the stress invariants as given in Appendix B. It has been verified by various checks (e.g. that the stress vector is tangential to the yield surface for all strain vectors) that this form of the stiffness matrix correctly represents the incremental plastic behaviour.

5. Conclusions

Plasticity models for modelling the behaviour of granular materials can be derived from thermodynamic functions. This approach has the advantage that the models are guaranteed to obey the laws of thermodynamics, and retrospective criteria need not be applied.

Within this framework a standard approach has been developed (subject only to a minor restriction on the form of the Dissipation Function) which allows the incremental stiffness matrix to be derived solely by manipulation of the original thermodynamic functions. This is an essential step if this approach is to be used to derive models appropriate for non-linear finite element analysis. Whilst the new approach needs further refinement, it can now readily be used for the development and implementation of realistic models for granular materials.
5. Acknowledgement

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6. References


7. Appendix A: An Alternative Formulation Using Internal Variables

In this Appendix the incremental response is derived from the original forms of the Helmholtz Free Energy and Dissipation Function, without introducing the definitions of elastic and plastic strains. On some occasions this approach proves to be more useful, since the internal variables are not restricted to be the plastic components of the strains. The numbering of the equations in this Appendix is chosen to correspond as closely as possible to the equations in the original text, so some numbers are omitted.

As in the main text, the dissipation function is not a function of the strain rates, so that equations (1) and (2) become:

\[
\sigma_i = F_i^e \tag{A1}
\]

\[
- F_i^{\alpha} = \frac{\partial D}{\partial \alpha_i} \tag{A2}
\]
Where the shorthand \( F_i^e = \frac{\partial F}{\partial \epsilon_i} \), \( F_{ij}^{ea} = \frac{\partial^2 F}{\partial \epsilon_i \partial \epsilon_j} \) etc. is introduced. Note that the dimension of the internal variable \( \alpha_i \) need not be the same as that of the strain \( \epsilon_i \). The differential of (A1) gives:

\[
\dot{\alpha}_i = F_{ij}^{ee} \dot{\epsilon}_j + F_{ij}^{ea} \dot{\alpha}_j
\]  \hspace{1cm} (A8)

Again assuming that \( D^2 \) is quadratic, define:

\[
q_{ij} = \frac{1}{2} \frac{\partial^2 (D^2)}{\partial \epsilon_i \partial \epsilon_j}
\]  \hspace{1cm} (A9)

Define also \( p_{ij} \), the inverse of \( q_{ij} \) as in the main text. Multiplying (A2) by \( \dot{\alpha}_i \) gives:

\[
- F_i^a \dot{\alpha}_i = \frac{\partial D}{\partial \epsilon_i} \dot{\alpha}_i = D
\]  \hspace{1cm} (A11)

Multiplying (A2) by \( D \) gives:

\[
- F_i^a D = \frac{\partial D}{\partial \epsilon_i} D = q_{ij} \dot{\alpha}_j
\]  \hspace{1cm} (A13)

or, defining \( r_j = - p_{ij} F_i^a \):

\[
- p_{ij} F_i^a D = r_j D = \dot{\alpha}_j
\]  \hspace{1cm} (A14)

The yield surface can therefore be derived as:

\[
p_{ij} F_i^a F_j^a = 1
\]  \hspace{1cm} (A15)

and the differential of the yield surface is:

\[
p_{ij} F_i^a F_j^a + 2p_{ij} F_i^a \left( F_{jk}^{ae} \dot{\epsilon}_k + F_{jk}^{aa} \dot{\alpha}_k \right) = 0
\]  \hspace{1cm} (A16)

This can be re-arranged to:

\[
r_{kl} \dot{q}_{kl} = -2r_j \left( F_{jk}^{ae} \dot{\epsilon}_k + F_{jk}^{aa} \dot{\alpha}_k \right)
\]  \hspace{1cm} (A17)
Which can be further be re-written as:

\[ r_k r_l \left( q_{klm}^e \dot{e}_m + q_{klm}^\alpha \dot{\alpha}_m \right) = -2 r_j \left( F_{jk}^{\alpha e} \dot{e}_k + F_{jk}^{\alpha \alpha} \dot{\alpha}_k \right) \] (A18)

where \( q_{klm}^e = \frac{\partial q_{kl}}{\partial e_m} \) etc., This allows a solution for \( D \):

\[ r_k r_l \left( r_n q_{klm}^\alpha + 2 F_{kl}^{\alpha \alpha} \right) D = -r_j \left( r_l q_{lj}^\alpha + 2 F_{lj}^{\alpha \alpha} \right) \dot{e}_j \] (A20)

which is again an equation of the form \( AD = B \dot{e}_j \), and one can derive:

\[ \sigma_j = \left( F_{ij}^{\alpha e} + \frac{\nu}{r_k} F_{jk}^{\alpha \alpha} \frac{B_j}{A} \right) \dot{e}_j = \delta_{ij} \dot{e}_j \] (A21)

A minor variation is that it often more convenient to express \( D \) in the form \( D = D(\sigma_i, \alpha_i, \dot{\alpha}_j) \), where the substitution of stress for strain variables is allowable because equation (A1) gives \( \sigma_j = \sigma_j(\epsilon_j, \alpha_i) \). The rest of the analysis proceeds exactly as above, except that (A18) becomes:

\[ r_k r_l \left( q_{klm}^\alpha \sigma_m + q_{klm}^{\alpha \alpha} \dot{\alpha}_m \right) = -2 r_j \left( F_{jk}^{\alpha \alpha} \dot{e}_k + F_{jk}^{\alpha \alpha} \dot{\alpha}_k \right) \] (A18a)

leading to:

\[ r_k r_l \left( q_{klm}^\alpha \sigma_m + q_{klm}^{\alpha \alpha} F_{kl}^{\alpha \alpha} \tau_n + 2 F_{kl}^{\alpha \alpha} \right) D = -r_j \left( r_l q_{lj}^\alpha F_{lj}^{\alpha e} + 2 r_k F_{kj}^{\alpha \alpha} \right) \dot{e}_j \] (A20a)

Which is again of the form \( AD = B \dot{e}_j \), and can be substituted into equation (A21).

8. Appendix B: Intermediate Results for Frictional Plasticity Model

The necessary intermediate results for the model developed in Section 4.3 are:

\[ F_i^e = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} K_v \frac{2G}{3} (2 \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - 2 \alpha_1 - \alpha_2) \\ K_v \frac{2G}{3} (-\varepsilon_1 + 2 \varepsilon_2 - \varepsilon_1 + \alpha_1 - \alpha_2) \\ K_v \frac{2G}{3} (-\varepsilon_1 - \varepsilon_2 + 2 \varepsilon_3 + \alpha_1 + 2 \alpha_2) \end{bmatrix} \] (B1)
\[ F_{ij}^{ee} = K \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{2G}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad F_{ij}^{ee} = \frac{2G}{3} \begin{bmatrix} -2 & -1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} \]  
(B3a,b)

\[ F_{ij}^{aa} = \frac{2G}{3} \begin{bmatrix} -2 & 1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad F_{ij}^{aa} = \frac{2G}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \]  
(B4a,b)

\[ q_{ij} = \frac{8 \mu^2}{9} \begin{bmatrix} \sigma_1 (\sigma_2 + \sigma_3) & \sigma_1 \sigma_3 \\ \sigma_1 \sigma_3 & \sigma_3 (\sigma_1 + \sigma_2) \end{bmatrix} \]  
(B5)

\[ p_{ij} = \frac{3}{8 \mu^2 \sigma_1 \sigma_2 \sigma_3} \begin{bmatrix} \sigma_3 (\sigma_1 + \sigma_2) & -\sigma_1 \sigma_3 \\ -\sigma_1 \sigma_3 & \sigma_1 (\sigma_2 + \sigma_3) \end{bmatrix} \]  
(B6)

\[ r_i = \frac{3}{8 \mu^2 \sigma_1 \sigma_2 \sigma_3} \begin{bmatrix} \sigma_3 (\sigma_1 - \sigma_2) \\ \sigma_3 (\sigma_2 - \sigma_3) \end{bmatrix} \]  
(B7)

\[ q_{ijk}^a = 0 \]  
(B8)

\[ q_{ijk}^a = \frac{8 \mu^2}{9} \begin{bmatrix} (\sigma_2 + \sigma_3) & \sigma_3 \\ \sigma_3 & \sigma_3 \end{bmatrix}, \quad q_{ijk}^b = \frac{8 \mu^2}{9} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_3 \end{bmatrix} \]  
(B9a,b)

\[ q_{ijk}^c = \frac{8 \mu^2}{9} \begin{bmatrix} \sigma_1 & \sigma_1 \\ \sigma_1 & (\sigma_1 + \sigma_2) \end{bmatrix} \]  
(B9c)

The manipulation of the above equations to obtain the stiffness matrix is tedious by hand, and has been carried out using the program Mathematica. Making use of the invariants $I_1 = \sigma_1 + \sigma_2 + \sigma_3$, $I_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1$, $I_3 = \sigma_1 \sigma_2 \sigma_3$, it can be derived that:

\[ A = \frac{GI_1 (2I_2^3 - 7I_1 I_2 I_3 + 9I_3^2)}{32 \mu^4 I_3^3} \]  
(B10)

The final form of the stiffness matrix is given in equation (11), in which the following three factors of the stress invariants are used: $C = 1 + \frac{I_2 (I_1 I_2^2 - 2I_1^2 I_3 - 3I_2 I_3)}{I_1 E}$, $D = \frac{3(I_1 I_2^2 - 2I_1^2 I_3 - 3I_2 I_3)}{I_1 E}$ and $E = 2I_1^3 - 7I_1 I_2 I_3 + 9I_3^2$.