The Work Input to an Unsaturated Granular Material

by

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In the development of constitutive relations for soils, care is needed in the choice of stress and strain variables so that they are properly chosen so as to be work conjugate, that is to say that the rate of input work (per unit volume) to the soil is equal to the sum of the product of the stresses with their corresponding strain rates. Houlsby (1979) showed that, under the approximation that both the soil grains and the pore fluid are incompressible, the rate of work input to a granular material which is simultaneously deforming and subject to fluid flow is given by the expression:

\[ L = \sigma_{ij}' \dot{\varepsilon}_{ij} - u_i' w_i \]  \( (1) \)

Where \( L \) is the power input per unit volume of soil, \( \sigma_{ij}' = \sigma_{ij} - u \delta_{ij} \) is the effective stress as conventionally defined, \( \dot{\varepsilon}_{ij} \) is the strain rate, \( u_i' \) is the excess pore pressure gradient and \( w_i \) is the artificial seepage velocity. The purpose of this Note is to extend the previous work to include unsaturated soils. The notation used here follows exactly that used by Houlsby (1979).

The \(-u_i' w_i\) term in equation (1) is simply the power dissipated due to the fluid flow, and the \( \sigma_{ij}' \dot{\varepsilon}_{ij} \) term demonstrates that the stresses which are work conjugate to the strains are simply the effective stresses. Since the mechanical behaviour of a soil depends on the way the power input is stored and dissipated, this means that it is the effective (not total) stresses which will change in response to straining of the soil. Equation (1) therefore provides the theoretical justification for the empirical definition of effective stress by Terzaghi (1943) that “it represents that part of the stress which produces measurable effects such as compaction or an increase of the shearing resistance”.

Houlsby (1981) showed that if the pore fluid is compressible there is an additional term \( n u_{ij}' (w) \) on the right hand side of (1), where \( n \) is the porosity and \( \dot{V}^{(w)} \) the volumetric strain in the pore fluid. This term is clearly interpreted as the power input required to compress the pore fluid.

In recent years there has been considerable interest in the understanding of the mechanical behaviour of unsaturated soils (see for instance Alonso, Gens and Josa (1990), Wheeler and Sivakumar (1995)). An important issue is the choice of the appropriate stress variables for such problems. Bishop (1959) suggested that the behaviour of unsaturated soils might be described in terms of a single effective stress defined as:

\[ \sigma_{ij}' = \sigma_{ij} - (u^{(a)} - \chi(u^{(a)} - u^{(w)})) \delta_{ij} \]  \( (2) \)

where \( u^{(a)} \) and \( u^{(w)} \) are the pore pressures in the air and water phases respectively and \( \chi \) is a factor which was expected to be a function of the saturation ratio \( S_r \). This approach has.

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however, been found to be unsatisfactory, in particular because it is incapable of modelling the process of compression (often in this context termed collapse) on wetting. Instead there is now a general consensus that it is necessary to describe the behaviour in terms of an additional independent variable as well as an appropriately defined effective stress. The most commonly accepted approach (introduced by Bishop and Blight (1963), see also Matyas and Radhakrishna (1968)) employs the “net stress” \( \sigma''_{ij} = \sigma_{ij} - u^{(a)} \delta_{ij} \) and the “suction” \( s = u^{(a)} - u^{(w)} \). A double prime notation is introduced for the net stress to avoid confusion between it and the effective stress.

It appears to be implicitly assumed, but nowhere proven, that the net stress is work conjugate to the strains. Furthermore there must be a strain-like quantity which is work conjugate to the suction, but this issue does not appear to have been explored. It is the purpose of this Note to examine these two linked issues. The following analysis follows closely the method used by Houlshby (1979), except that the compatibility relationships are introduced at a later stage in the analysis.

First it is necessary to set up a number of definitions. In an unsaturated soil in which there are different pore water and pore air pressures, this arises because of surface tension and the curvature of the air-water menisci. In expressing the stress in terms of its component parts in the solid, water and air phases, it is necessary therefore to include a further term which represents the summed effect of the surface tensions. Fredlund and Morgenstern (1977) in fact justify the introduction of the air-water interface (which they term the “contractile skin”) as a fourth phase. It is sufficient for our purposes here that the forces in this contractile skin are represented in an averaged way by a stress-like term \( T_{ij} \).

If the porosity is \( n \) then note that the volume fractions of solids, water and air are \( (1-n) \), \( S_r u \) and \( (1 - S_r) n \) respectively. If the average stress in the soil grains is \( s_{ij} \), then the total stress is:

\[
\sigma_{ij} = n(S_r u^{(w)} + (1 - S_r) u^{(a)}) \delta_{ij} + (1 - n) s_{ij} + T_{ij}
\]  

(3)

The overall density is given by:

\[
\rho = n(S_r \rho^{(w)} + (1 - S_r) \rho^{(a)}) + (1 - n) \rho^{(s)}
\]  

(4)

where \( \rho^{(s)} \), \( \rho^{(w)} \) and \( \rho^{(a)} \) are the densities of the solid, water and air phases. The excess pore pressure gradients in the two fluid phases are defined by:

\[
u_{ij}^{(w)} = u_{ij}^{(w)} - \rho^{(w)} g_i
\]

(5)

\[
u_{ij}^{(a)} = u_{ij}^{(a)} - \rho^{(a)} g_i
\]

(6)

where \( g_i \) is the gravitational acceleration vector and a comma notation is used to indicate differentiation with respect to spatial co-ordinate. The (compressive positive) strain rate is given by:
\[ \dot{\varepsilon}_{ij} = -v_{(j,i)} \]  \hspace{2cm} (7)

where a superposed dot indicates the time differential, \( v_{j} \) is the velocity vector for the solids and \( t_{(ij)} \) indicates the symmetric part of the tensor \( t_{ij} \).

If \( f_{i}^{(w)} \) and \( f_{i}^{(a)} \) are the average velocities of the water and air phases, then the artificial seepage velocities (the velocities used in conventional flow calculations) of the water and air are defined by:

\[ w_{i}^{(w)} = nS_{r}\left(f_{i}^{(w)} - v_{i}\right) \quad \hspace{2cm} (8) \]
\[ w_{i}^{(a)} = n\left(1 - S_{r}\right)\left(f_{i}^{(a)} - v_{i}\right) \quad \hspace{2cm} (9) \]

Total stress equilibrium requires that:

\[ -\sigma_{ij,j} + \rho g_{i} = 0 \quad \hspace{2cm} (10) \]

The power input to a volume \( V \) fixed in space, with a bounding area \( A \), is the sum of the power input at the boundary and the power done by gravitational forces:

\[
\int_{V} LdV = -\int_{A} \left[ nS_{r}u^{(w)}f_{i}^{(w)} + (1 - S_{r})u^{(a)}f_{i}^{(a)} \right] \delta_{ij} + (1 - n)S_{r}v_{i} + T_{ij}v_{j}^{(r)} \right] n_{j} dA \\
\quad + \int_{V} \left[ nS_{r}\rho^{(w)}f_{i}^{(w)} + (1 - S_{r})\rho^{(a)}f_{i}^{(a)} \right] (1 - n)\rho^{(c)}v_{i} \right] g_{i} dV
\quad \hspace{2cm} (11) \]

where \( n_{i} \) is the outward normal to the surface \( A \), and the negative sign on the area integral arises from the compressive positive convention. The variable \( v_{i}^{(r)} \) is the velocity of the contractile skin (the air-water menisci), which in principle does not need to be equal to either of the velocities of the air or water themselves.

Substituting for stress and density gives:

\[
\int_{V} LdV = -\int_{A} \left[ nS_{r}u^{(w)}(f_{i}^{(w)} - v_{i}) + (1 - S_{r})u^{(a)}(f_{i}^{(a)} - v_{i}) \right] \delta_{ij} + \sigma_{ij}v_{i} + T_{ij}(v_{j}^{(r)} - v_{i}) \right] n_{j} dA \\
\quad + \int_{V} \left[ nS_{r}\rho^{(w)}(f_{i}^{(w)} - v_{i}) + (1 - S_{r})\rho^{(a)}(f_{i}^{(a)} - v_{i}) \right] + \rho v_{i} \right] g_{i} dV
\quad \hspace{2cm} (12) \]

and further substituting the artificial velocity equations gives:
\[
\int_V LdV = -\int_A \left[ u^{(w)} w_j^{(w)} + u^{(a)} w_j^{(a)} + \sigma_{ji} v_j + T_{ij}(v_i^{(v)} - v_i) \right] v_j dA \\
+ \int_V \left[ p^{(w)} w_i^{(w)} + p^{(a)} w_i^{(a)} + \rho v_i \right] g_i dV
\] (13)

Applying the divergence theorem of Gauss then leads to:

\[
\int_V LdV = \int_V \left[ u^{(w)} w_j^{(w)} + u^{(a)} w_j^{(a)} + \sigma_{ji} v_j + T_{ij}(v_i^{(v)} - v_i) \right]_j + \left[ p^{(w)} w_i^{(w)} + p^{(a)} w_i^{(a)} + \rho v_i \right] g_i dV
\] (14)

In most cases of interest, either the air phase (for high \( S_r \) values) or the water phase (for low \( S_r \) values) is likely to be static with respect to the soil grains. In either of these cases it is therefore expected that (on average) the on average the contractile skin will move with the soil skeleton, so that \( v_i^{(v)} - v_i = 0 \). It will be assumed in the following that this is the case, so that the term in \( T_{ij} \) above can be dropped. The full implications of this assumption are yet to be explored.

Noting that the volume \( V \) is arbitrary, it follows that the integrands on both sides of the equation are equal. Expanding the right hand side (and omitting the term in \( T_{ij} \)) then leads to:

\[
L = -u_j^{(w)} w_j^{(w)} - u_j^{(a)} w_j^{(a)} - \sigma_{ji} v_j - u^{(w)} w_{j,j} - u^{(a)} w_{j,j} - \sigma_{ji} v_i \\
+ p^{(w)} w_i^{(w)} g_i + p^{(a)} w_i^{(a)} g_i + \rho v_i g_i
\] (15)

Introducing the equilibrium equation and the excess pore pressure gradient definitions allows simplification to:

\[
L = -u_j^{(w)} w_j^{(w)} - u_j^{(a)} w_j^{(a)} - u^{(w)} w_{j,j} - u^{(a)} w_{j,j} - \sigma v_i v_i
\] (16)

The compatibility relationships are significantly more complex for the three-phase problem than for saturated soil. They are best expressed by noting that for each phase the rate of mass outflow from a surface of a volume fixed in space is equal to the rate of reduction of mass in that volume:

\[
\int_A \rho^{(s)} (1-n)v_i n_i dA = -\int_V \frac{d}{dt} (\rho^{(s)} (1-n)) dV
\] (17)

\[
\int_A \rho^{(w)} S_j n_{j_i}^{(w)} n_i dA = -\int_V \frac{d}{dt} (\rho^{(w)} S_j n) dV
\] (18)

\[
\int_A \rho^{(a)} S_j n_{j_i}^{(a)} n_i dA = -\int_V \frac{d}{dt} (\rho^{(a)} S_j n) dV
\] (19)
Applying the divergence theorem, noting that (under the assumption of the incompressibility of the solids and water) both \( \rho^{(i)} \) and \( \rho^{(w)} \) are constant, and again making use of the fact that \( \dot{V} \) is arbitrary, the above lead to:

\[
\mathbf{v}_{i,j} = \frac{\dot{n}}{1-n} \tag{20}
\]

\[
f_{i,j}^{(w)} = -\frac{\dot{S}_r}{S_r} - \frac{\dot{n}}{n} \tag{21}
\]

\[
f_{i,j}^{(a)} = \frac{\dot{S}_r}{(1-S_r)} - \frac{\dot{n}}{n} - \frac{\dot{\rho}^{(a)}}{\rho^{(a)}} \tag{22}
\]

In which it has been assumed that spatial variations of \( n \) and \( S_r \) are insignificant. Substituting the artificial velocities and noting that \( \mathbf{v}_{i,j} = -\dot{\varepsilon}_{ii} \) leads to:

\[
w_{i,j}^{(w)} = -n\dot{S}_r + S_r\dot{\varepsilon}_{ii} \tag{23}
\]

\[
w_{i,j}^{(a)} = n\dot{S}_r + (1-S_r)\dot{\varepsilon}_{ii} - n(1-S_r)\frac{\dot{\rho}^{(a)}}{\rho^{(a)}} \tag{24}
\]

Substituting (23) and (24) into (16), and noting that \( -\sigma_{ij} \mathbf{v}_{i,j} = \sigma_{ij} \dot{\varepsilon}_{ij} \), it follows that:

\[
L = -u_{j,j}^{(w)}w_{j,j}^{(w)} - u_{j,j}^{(a)}w_{j,j}^{(a)} - n(u^{(a)} - u^{(w)})\dot{S}_r - (S_r u^{(w)} + (1-S_r)u^{(a)})\dot{\varepsilon}_{ij} + n(1-S_r)u^{(a)}\frac{\dot{\rho}^{(a)}}{\rho^{(a)}} + \sigma_{ij} \dot{\varepsilon}_{ij} \tag{25}
\]

which further simplifies to:

\[
L = -u_{j,j}^{(w)}w_{j,j}^{(w)} - u_{j,j}^{(a)}w_{j,j}^{(a)} - nS_r\dot{\varepsilon}_{ij} + n(1-S_r)u^{(a)}\dot{\varepsilon}_{ij} + \left(\sigma_{ij} - (S_r u^{(w)} + (1-S_r)u^{(a)})\delta_{ij}\right)\dot{\varepsilon}_{ij} \tag{26}
\]

where \( \dot{\varepsilon}_{ij}^{(a)} = \frac{\dot{\rho}^{(a)}}{\rho^{(a)}} \) is the volumetric strain rate of the air phase.

The interpretation of equation (26) is as follows. The first two terms are the power dissipated by the flow of water and air through the soil. The third term demonstrates that the strain-like quantity that is work conjugate to the suction \( s \) is the quantity \(-nS_r\). The fourth term is the power input to compress the air phase, and the fifth term demonstrates that the quantity which is work-conjugate to the strains is the effective stress defined as:

\[
\sigma_{ij}' = \sigma_{ij} - (S_r u^{(w)} + (1-S_r)u^{(a)})\delta_{ij} \tag{27}
\]
which is simply Bishop’s expression with \( \chi = S_r \). Note that this expression conveniently reduces to the conventional expressions for effective stress for the two cases of fully saturated and completely dry soils.

The above does **not** imply that unsaturated soils can be described simply by adopting Bishop’s original approach together with \( \chi = S_r \), since it acknowledges that the mechanical behaviour of the soil is also governed by the suction \( s \). The correct interpretation is that the simplest choice of stress and strain variables for unsaturated soils would be an effective stress \( \sigma'_i = \sigma_i - \left( S_r u^{(w)} + (1 - S_r) u^{(a)} \right) \delta_i \) with conjugate strain rate \( \dot{\varepsilon}_i \) and a modified suction (scaled by the porosity) \( ns \) with conjugate “strain rate” \( -\dot{S}_r \). However, any linear combination of these stress variables is acceptable, provided that the appropriate changes are also made to the strain rate definitions. Noting therefore that:

\[
ns(-\dot{S}_r) + \left( \sigma_i - \left( S_r u^{(w)} + (1 - S_r) u^{(a)} \right) \delta_i \right) \dot{\varepsilon}_i = s(-n\dot{S}_r + S_r \dot{\varepsilon}_r) + \left( \sigma_i - u^{(a)} \delta_i \right) \dot{\varepsilon}_i
\]

(28)

it would also be acceptable to use the net stress \( \sigma'' = \sigma_i - u^{(a)} \delta_i \) with conjugate strain rate \( \dot{\varepsilon}_i \) and the suction \( s \) with conjugate strain rate \( -n\dot{S}_r + S_r \dot{\varepsilon}_r \). The latter can be shown to be equal to \(-\frac{\dot{v}_w}{v}\), where \( v \) is the specific volume \((1 + e)\) and \( v_w \) is the quantity \((1 + S_r e)\) defined by Wheeler (1991). Wheeler (1995) correctly states that \(-\frac{\dot{v}_w}{v}\) is the strain that is conjugate to \( s \), but does not provide a proof. The advantages, however, of using \( \sigma'_i = \sigma_i - \left( S_r u^{(w)} + (1 - S_r) u^{(a)} \right) \delta_i \) and \( ns \) are that, firstly the effective stress definition is entirely consistent with the conventional approach both for fully saturated and completely dry soils, and secondly that \( ns \) is associated with the degree of saturation, which is more simply defined than \(-\frac{\dot{v}_w}{v}\).

This note has achieved two purposes. Firstly an appropriate form of the effective stress which is work conjugate to the strains for an unsaturated soil has been obtained. Secondly the strain-like quantity conjugate to the suction has been identified. This latter result will allow formulation of complete models for unsaturated soils including prediction of changes of saturation, which is an element missing from most current models. These results have been obtained under the approximation that the solid and water phases may be regarded as incompressible, and under a simplifying assumption about the movement of air–water menisci. Certain other implicit assumptions about the validity of averaging processes are also inherent in the analysis.

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References


