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FINITE ELEMENTS FOR MODELLING MATERIAL
INCOMPRESSIBILITY USING EXACT INTEGRATION

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SUITABILITY OF TWO AND THREE DIMENSIONAL FINITE ELEMENTS FOR MODELLING MATERIAL INCOMPRESSIBILITY USING EXACT INTEGRATION

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SUMMARY

This paper examines the suitability of two and three dimensional finite elements to model accurately problems involving material incompressibility, using the displacement finite element method and exact numerical integration. The previously used method for classification of element suitability is considered not to provide a rigorous quantitative measure of element suitability, nor to allow comparisons between elements of different order and type, when computational effort is a consideration. Hence three new parameters, quantified in terms of free degrees-of-freedom (equal to the degrees-of-freedom minus the incompressibility constraints), are introduced. These are used to examine triangular and Serendipity quadrilateral two dimensional finite elements, and tetrahedral, Serendipity cubic and Lagrangian cubic three dimensional elements configured in a regular cubic arrangement. The findings of this paper are substantiated by a number of three-dimensional numerical experiments and comparison with a separate two-dimensional study.

For two dimensional plane strain conditions, most low order triangular and quadrilateral elements are suitable for analysis involving material incompressibility, however, some elements give a better computational efficiency than others. The linear strain triangle is thought to be the most efficient computationally for most plane strain applications. For axisymmetric conditions, the 15-node cubic strain triangle is the lowest order suitable element. For three dimensional analysis, all Serendipity cubic elements are found to be unsuitable. The quadratic strain tetrahedron and higher order tetrahedra are suitable when placed in an arrangement of either a 5 or 6 tetrahedra per cube. The linear strain tetrahedron is on the border-line of suitability and is thought to be only suitable if the mesh boundary nodes are not over-constrained. The Lagrangian cubic elements of higher order than the 27-node cube are suitable, but are not as efficient computationally as the corresponding tetrahedral elements.
1. INTRODUCTION

In the analysis of incompressible materials using the displacement finite element method, an over-stiff response has been observed in many cases. This response has been observed in elastic-plastic problems where incompressibility applies in the fully plastic region (Nagtegaal et al, 1974; Sloan and Randolph, 1982; de Borst and Vermeer, 1984) or problems where incompressibility is enforced point-wise throughout the continuum (Sloan and Randolph, 1982; Burd and Houlsby, 1990). The inaccuracy of the load-displacement curve is a result of the incompressibility condition imposing additional kinematic constraints on the nodal displacements, thereby reducing the available "free" degrees-of-freedom in the finite element mesh.

Nagtegaal et al (1974), in their pioneering work in this area, proposed a criterion for determining the suitability of a particular finite element for elastic-plastic analysis. They evaluate the ratio of the total number of degrees-of-freedom (equal to the number of nodes multiplied by the number of degrees-of-freedom per node) to the total number of constraints (the number of elements multiplied by the number of incompressibility constraints per element). This ratio is evaluated in the limit as the mesh is refined, assuming a special arrangement of elements is not employed1. If this ratio is greater than one, the finite element is suitable for material incompressibility analysis. Similarly if the ratio is less than one, the finite element is unsuitable. However, there is some conjecture about the suitability of finite elements when the ratio is equal to one, and this issue is addressed in this paper.

With the acknowledgement that certain finite elements yield inaccurate results in analyses where the material is (or becomes) incompressible, much effort has been given to developing special finite element formulations in which the detrimental effect of incompressibility is reduced. A concise summary of the many formulations developed is provided by Burd and Houlsby (1990). The methods include techniques such as uniform reduced integration, reduced selective integration, and mixed finite element formulations. Each method has advantages and disadvantages and no single technique has emerged as a clear first choice. A different philosophy to many of those developed was introduced by Sloan and Randolph (1982). They demonstrated that as the order of the polynomial defining the displacements within an element is increased, the increase of new degrees-of-freedom is greater than the increase in the incompressibility constraints. Therefore if a particular type of finite element has a ratio of degrees-of-freedom to constraints of less than one, then an element suitable for incompressibility analysis may be found by progressing to a higher order element of the same family.

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1 Special arrangements of elements are discussed in section 6.
The main objective of this paper is to document and compare the suitability of the Lagrangian rectangular prism (hereafter referred to as "cube"), Serendipity cube and tetrahedron three dimensional finite element families, for the analysis of incompressible materials using the displacement finite element method and exact numerical integration of the stiffness matrices. The documentation of element suitability is initially performed using the ratio of degrees-of-freedom to incompressibility constraints, as described by Nagtegaal et al (1974). However, to facilitate a more quantitative understanding of the problem and develop a comparative measure of performance applicable to finite elements of different type and order, two new parameters are introduced. These are applied to the two-dimensional elements previously examined by Sloan and Randolph (1982) and the three-dimensional elements mentioned above. The important consideration of computational effort is then introduced into the comparison of the finite elements examined using a further new parameter which is a direct measure of the computational performance of a given finite element. Two three-dimensional numerical experiments are then reported and compared with a similar two-dimensional study of Sloan (1981). These help to clarify and substantiate the findings of the previous sections.

2. THE RATIO OF DEGREES-OF-FREEDOM TO INCOMPRESSIBILITY CONSTRAINTS FOR THREE DIMENSIONAL FINITE ELEMENTS

Sloan and Randolph (1982), following the work of Nagtegaal et al (1974), demonstrated that the ratio of total number of degrees-of-freedom to total number of constraints, in the limit as the mesh is uniformly refined, can be evaluated at an element level. That is

\[
\frac{\text{degrees-of-freedom}}{\text{constraints}} = \frac{\text{degrees-of-freedom per element}}{\text{constraints per element}}
\]  

(1)

For a two dimensional finite element mesh, the ratio of number of nodes \((n)\) to number of elements \((e)\) in the limit as the mesh is uniformly refined is (Nagtegaal et al, 1974; Sloan and Randolph, 1982)

\[
n^* = \lim_{n \rightarrow \infty} \left( \frac{n}{e} \right) = \frac{m}{2}
\]

(2)

where \(m\) is the sum of the internal nodal (planar) angles divided by \(\pi\). The notation adopted throughout this paper is an asterix (*) refers to the limit as the mesh is uniformly refined \((n \rightarrow \infty)\) and the subscript \(e\) represents the division by the number of elements. Noting there are two degrees-of-freedom per node, Equation (2) can be written as

\[
n^* = \lim_{n \rightarrow \infty} \left( \frac{d}{e} \right) = m
\]

(3)
where the number of degrees-of-freedom per element \( d' \) is directly related to the sum of the internal nodal angles represented by \( m \). The proof of this relationship will not be repeated here, but its extension to three dimensional problems is as follows.

Consider a regular 27-node Lagrangian cubic element as shown in Figure 1b. This element consists of 8 corner nodes, 12 edge nodes, 6 face nodes and 1 interior node. In direct analogy to planar angles, the "volumetric" internal angles of these nodes (measured in steradians) are \( \pi/2 \) (corner node A), \( \pi \) (edge node B), \( 2\pi \) (face node C) and \( 4\pi \) (interior node D), respectively\(^2\). The sum of the internal volumetric nodal angles is therefore 32\( \pi \). For any general three dimensional finite element, the sum of the internal volumetric nodal angles can be written as

\[
\sum_{i=1}^{l} \theta_i = m\pi
\]

(4)

where \( l \) is the number of nodes per element and \( m \) varies with element type (and the configuration of the elements, as will be shown later). For a regular mesh of \( e \) elements, it therefore follows the sum of the mesh internal nodal volumetric angles is

\[
\sum_{j=1}^{e} \sum_{i=1}^{l} \theta_i = em\pi
\]

(5)

An alternative way of summing the internal nodal angles of the mesh is to consider the nodes alone without any reference to elements. For the purpose of simplicity, let us assume that the mesh is a rectangular prism with \( n \) nodes consisting of 8 corner nodes, \( r \) edge nodes, \( s \) face nodes and \( k \) interior nodes. Noting the internal volumetric nodal angles for the 27-node Lagrangian cube discussed earlier, it follows that

\[
\sum_{j=1}^{e} \sum_{i=1}^{l} \theta_i = (4 + r + 2s + 4k)\pi
\]

(6)

Equating (5) and (6) and rearranging gives

\[
\frac{e}{k} = \frac{4}{m} + \frac{(4 + r + 2s)}{mk}
\]

(7)

\(^2\)Nodes A, B, C, and D are shown in Figure 1b
Upon uniform refinement of the mesh, $k$ will increase at a greater rate than the number of boundary nodes $(8 + r + s)$ and therefore will tend to the total number of nodes $(n)$. In this limit, the second term on the right side of (7) becomes small and the ratio of number of nodes to number of elements becomes

$$n^*_e = \lim_{k \to \infty} \left( \frac{k}{e} \right) = \frac{m}{4} \quad (8)$$

Rewriting this in terms of degrees-of-freedom per element $(d^*_e)$ for a three dimensional mesh gives

$$d^*_e = \lim_{n \to \infty} \left( \frac{d}{e} \right) = \frac{3}{4}m \quad (9)$$

Hence, in the limiting case of a uniformly refined mesh, the degrees-of-freedom per element for a three dimensional finite element is three-quarters the sum of the element internal nodal volumetric angles divided by $\pi$ ($m$). For a mesh with any given external boundary configuration, the number of interior nodes $(k)$ will always tend to the total number of nodes $(n)$ in the limiting case of a uniformly refined mesh. And (8) and (9) will similarly result, independent of the rectangular prismatic mesh adopted in the above formulation.

Unlike the two dimensional case, where the value of $m$ is constant for a given element arranged randomly in a mesh, for three dimensional elements, $m$ is dependent on the arrangement of the elements since the sum of the internal volumetric angles of a polyhedron is not constant (Nagtegaal et al, 1974). Therefore the degrees-of-freedom per element for three dimensional finite elements is dependent on the arrangement of the elements adopted. All three dimensional element configurations examined in this paper are of a regular (orthogonal) cubic arrangement. This type of arrangement is commonly used in three dimensional finite element analysis because it simplifies mesh generation and visualisation.

The procedure for evaluating the number of constraints per element $(c_e)$ for plane strain and axisymmetric two dimensional conditions has been well documented by Nagtegaal et al (1974) and Sloan and Randolph (1982). Extension to three dimensional problems is straightforward, as briefly summarised below.

Under incompressible conditions the element is required to deform so that the volumetric strain rate $\dot{e}_{kk} = 0$. For a three dimensional problem, this condition may be written
\[
\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy} + \dot{\epsilon}_{zz} = \frac{\partial \dot{u}}{\partial x} + \frac{\partial \dot{v}}{\partial y} + \frac{\partial \dot{w}}{\partial z} = 0
\]  

(10)

where \( \dot{u} \), \( \dot{v} \) and \( \dot{w} \) denote velocities in the x, y and z directions, respectively. Within each finite element these can be expressed in the vector form

\[
\dot{\mathbf{u}} = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \mathbf{a} + \mathbf{b} x + \mathbf{c} y + \mathbf{d} z + \ldots
\]  

(11)

where the vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \ldots \), are expressed in terms of the nodal velocities and coordinates. Substituting Equation (11) into (10) yields an equation in terms of the coefficients of the \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \ldots \) vectors. Assuming "exact" integration is to be used in the stiffness formulation of the element, the number of constraints for the element (\( c_e \)) is taken to be equal to the number of coefficients (or combinations of coefficients) which are constrained to be zero in the resulting equation.

As addressed by Sloan and Randolph (1982), it is usual in finite elements to evaluate the element stiffness formulations at a discrete number of integration points. For a given straight-sided element, there is a particular number and positioning of the integration points such that the element stiffness is determined exactly. This is termed exact integration (Laursen and Gellert, 1978). The incompressibility constraints imposed on an element vary with the number of integration points. The approach of reducing the number of constraints by reducing the number of integration points and thereby improving the suitability of a particular element for incompressibility analysis has been widely used and is called "reduced integration" (Zienkiewicz, 1977). However, it has been shown that under certain conditions this inexact method may be unacceptably inaccurate (de Borst and Vermeer, 1984; de Borst, 1982). All reference to incompressibility constraints made in this paper assumes exact integration of a straight-sided element.

Prior to examining the suitability of some common three dimensional finite elements for incompressible analysis based on the analyses described previously, it is appropriate to rewrite Equation (1) in terms of the notation adopted in this section

\[
d_c^* = \frac{d_c^*}{c_e}
\]  

(12)

where \( d_c^* \) is the ratio of the degrees-of-freedom to constraints in the limit of a uniformly refined mesh.
2.1 Lagrangian Cube Family

The first four elements of the Lagrangian cube family, ranging from the 8-node cube to the 125-node cube, are given in Figure 1. Each element is defined by the parameter $N$, the order to which the polynomial interpolation displacement function is complete. For example, the displacement function for the 8-node cube is complete to order one ($N=1$). Furthermore, the number of nodes per element, and therefore number of polynomial displacement terms per element, is $(N + 1)^3$.

Table 1 demonstrates the suitability of the Lagrangian cubic elements (Figure 1) for incompressible material analysis using the displacement finite element method. The first two elements of this family, the 8-node and 27-node cubes, are shown to be unsuitable since their ratios of degrees-of-freedom to constraints ($d^*$) are less than one. However, $d^*$ is seen to increase with the order of the polynomial displacement function ($N$), this result being consistent with the findings of Sloan and Randolph (1982) for two dimensional finite elements. Subsequent higher order elements to the 27-node cube have a $d^*$ value greater than one, and are therefore classified as suitable.

Closer examination of Table 1 reveals that a relationship exists between the parameter $N$ and the degrees-of-freedom per element ($d_0^*$) and constraints per element ($c_*$). These are as follows

$$d_0^* = 3N^3 \quad (13)$$

$$c_* = (N + 1)^3 - 1 \quad (14)$$

Equations (13) and (14) combined with Equation (12) can be used to extend Table 1 to any higher order Lagrangian cubic element so desired.

2.2 Serendipity Cube Family

The suitability for accurate incompressible material analysis of the first three elements of the Serendipity cube family, schematically presented in Figure 2, is presented in Table 2. All three elements have values of $d_0^*$ of less than one and are therefore classified as unsuitable. These results are consistent with the findings of Nagtegaal et al (1974) who examined the suitability of the first two elements of this family. Note that the 8-node cube is equivalent to the 8-node Lagrangian cubic element and hence shows the same result.

Extension of the analysis to the Serendipity element of order 4 was attempted without success. The difficulty lies in that the choice of polynomial terms of the displacement function and
distribution of face and internal element nodes for this element is ambiguous. It appears that
to have a displacement function complete to order four requires 3 face or interior nodes to be
introduced, which cannot be achieved symmetrically.

2.3 Tetrahedron Family

Table 3 demonstrates the suitability for incompressibility analysis of the first four elements of
the tetrahedron family (Figure 3) when arranged in a regular cubic type lattice. However, the
tetrahedron family of elements is complicated by the fact that a cube can be divided into either
5 tetrahedra, as shown in Figure 4a, or 6 tetrahedra, as given in Figure 4b. As stated earlier, for
three dimensional problems the ratio of degrees-of-freedom to elements in the limit of a
uniformly refined mesh is dependent on the configuration of the elements. Therefore
arrangements of both 5 and 6 tetrahedra cube are considered in Table 3. The results for the 6
tetrahedra cube arrangement are also applicable to a triangular prism divided into 3 tetrahedra
(Figure 4c), since the former (Figure 4b) is simply the addition of two of the latter.

The 4 node constant strain tetrahedron has values of $d^*_c$ of less than one for both 5 and 6 tetrahedra
per cube configurations and is therefore unsuitable for incompressibility analysis. However,
the linear strain tetrahedron is suitable when placed in a 5 tetrahedra per cube arrangement ($d^*_c = 1.05$) and is on the borderline of suitability for the 6 tetrahedra per cube arrangement ($d^*_c = 1$). For the 5 tetrahedra arrangement, the results for the constant and linear strain tetrahedra
are consistent with the findings of Nagtegaal et al (1974). The higher order tetrahedral elements
in Table 3 have values of $d^*_c$ well above one for both configurations and are therefore suitable.

As for the Lagrangian cube family, relationships exist between $N$ and the degrees-of-freedom
per element and constraints per element. These can be used to extend Table 3 to higher order
elements. The relationship for the constraints per element ($c_e$) is

$$c_e = \frac{N(N+1)(N+2)}{6}$$

(15)

The relationships for the 5 and 6 tetrahedra cube arrangements between $N$ and the
degrees-of-freedom per element ($d^*_c$) are (noting that a unique relationship for the 5 tetrahedra
per cube arrangement only exists for those elements of order 4 or higher)

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3A discussion on the suitability of elements when $d^*_c = 1$ is given in section 6.

4Tetrahedral elements have the unique characteristic that the polynomial interpolation function describing the
nodal displacements is "complete" to order N, that is, no additional polynomial terms exist.
5 tetrahedra per cube:

\[ d^*_e = \frac{1}{10}N(5N^2 + 1) \quad \text{for} \quad N \geq 4 \]  

(16)

6 tetrahedra per cube:

\[ d^*_e = \frac{1}{2}N^3 \]  

(17)

3. THE COMPARATIVE PERFORMANCE OF FINITE ELEMENTS IN TERMS OF FREE DEGREES-OF-FREEDOM

While considering the ratio of degrees-of-freedom to constraints \( d^*_e \), in the limit as the mesh is uniformly refined, provides a qualitative measure on the suitability of an element, it does not provide a rigorous quantitative measure of element suitability. A more meaningful approach to the problem is to consider the free degrees-of-freedom per element \( f^*_e \) defined as

\[ f^*_e = d^*_e - c_e \]  

(18)

This new parameter defines the number of free degrees-of-freedom introduced by each element after the constraints imposed by the incompressibility condition are taken into account. There is of course a direct relationship between \( f^*_e \) and \( d^*_e \). An element is taken to be suitable for the analysis of an incompressible material, using the displacement finite element method, if \( f^*_e \) is greater than zero since each element then introduces free degrees-of-freedom to the problem, and so allows an accurate prediction of the load-displacement curve to be approached. Similarly if \( f^*_e \) is less than zero, the element is unsuitable since no free degrees-of-freedom are introduced and the mesh is over-constrained. Based on the definition of \( f^*_e \), one would also expect an element to be unsuitable when \( f^*_e \) is equal to zero. That is, if one considers a sub-region of the mesh away from the boundaries, then the available degrees-of-freedom will be governed solely by the parameter \( f^*_e \).

However, \( f^*_e = 0 \) is equivalent to the parameter \( d^*_e = 1 \), which Sloan and Randolph interpret to correspond to a suitable element. Sloan and Randolph (1982) base this interpretation on their examination of two dimensional mesh suitability when incompressibility is enforced throughout the mesh. They present an inequality which provides the required number of unrestrained boundary degrees-of-freedom, such that the total number of degrees-of-freedom will exceed the total number of boundary and incompressibility constraints. Based on this criterion for mesh suitability, they show that an element with \( d^*_e = 1 \) is suitable if the number of unrestrained mesh boundary degrees-of-freedom exceeds the number of restrained boundary degrees-of-freedom. They support their argument by presenting accurate finite element collapse load results for a smooth rigid strip footing (in plane strain) using the 8-node and 12-node quadrilateral elements,
which have a value of $d_e^* = 1$. The discrepancy between what the free degrees-of-freedom philosophy predicts and that postulated and observed by Sloan and Randolph (1982) is addressed later in the discussion of section 6.

While $f_e^*$ does help to quantify the suitability of an element, it does not allow comparison between different element types since the number of elements in a mesh with a given number of nodes will vary with element type. For the purpose of comparison it is better to normalise the number of free degrees-of-freedom with respect to the number of nodes, rather than the number of elements. This introduces the new parameter of free degrees-of-freedom per node ($f_n^*$) which takes into account the number of nodes per element ($n_e^*$) in the limit of a uniformly refined mesh.

$$f_n^* = \lim_{n \to \infty} \left( \frac{f_e^*}{n_e^*} \right) = \lim_{n \to \infty} \left( \frac{f_e^*}{n_e^*} \right) = \frac{d_e^* - c_e}{n_e^*}$$  \hspace{1cm} (19)

### 3.1 Two-dimensional Finite Elements

For the two-dimensional problem, substituting Equation (3) into Equation (18) gives the number of free degrees-of-freedom per element

$$f_e^* = m - c_e$$  \hspace{1cm} (20)

and substituting Equations (2) and (3) into Equation (19) gives the number of free degrees-of-freedom per node

$$f_n^* = 2 - 2 \frac{c_e}{d_e^*} = 2 - 2 \frac{c_e}{m}$$  \hspace{1cm} (21)

Table 1 from Sloan and Randolph (1982), which examines the suitability of the two-dimensional triangular and Serendipity quadrilateral finite elements for plane strain and axisymmetric conditions, is repeated in Table 4. The parameters $f_e^*$ and $f_n^*$ introduced in this paper are included in the Table. For plane strain conditions, the linear, quadratic and cubic strain triangular elements and the 17-node quadrilateral element are shown to introduce free degrees-of-freedom under incompressibility conditions and are therefore suitable. The constant strain triangle and the 8-node and 12-node quadrilateral elements have values of $f_e^* = 0$ and therefore are on the border-line of suitability.
For axisymmetric conditions, Table 4 shows the lowest order suitable element to be the cubic strain triangle, this being consistent with the findings of Sloan and Randolph (1982). However, an alternative approach to the problem has been developed by Yu (1990). By employing a modified displacement interpolation function, he showed the 6-node triangular element satisfies the incompressibility condition for the axisymmetric incompressibility condition. This formulation stems from the fact that the standard 6-node linear strain triangle is suitable under plane strain conditions (Table 4).

The parameter free degrees-of-freedom per node $f^*_n$, defined in Equation (18) and given in Table 4, can be used to compare the performance of different suitable finite elements. The comparison can be made in the following way. Consider the ratio of the number of nodes between meshes of different element types such that the same number of free degrees-of-freedom are introduced. It is expected that this will lead to calculations of similar accuracy. For example from Table 4, a 17-node quadrilateral mesh ($f^*_n = 1/4$) would require twice as many nodes as a linear strain triangle mesh ($f^*_n = 1/2$) to give approximately the same accuracy.

The above discussion, however, is limited in that the parameter $f^*_n$ does not take into consideration the computational effort involved for different element types and meshes of different sizes (number of nodes). This important detail is addressed in section 4.

3.2 Three-dimensional Finite Elements

For the three-dimensional problem, substituting Equation (9) into Equation (18) gives the number of free degrees-of-freedom per element

$$f^*_e = \frac{3}{4}m - c_e$$

Substituting Equations (8) and (9) into Equation (19) gives the number of free degrees-of-freedom per node

$$f^*_n = 3 - \frac{3c_e}{d_e} = 3 - \frac{4c_e}{m}$$

The parameters from Equations (22) and (23) are included in Tables 1 to 3 for the Lagrangian cubic, Serendipity cubic and tetrahedral, elements, respectively. It is interesting to note in Table 3 that for the quadratic and cubic tetrahedral elements, $f^*_n$ values for the 5 and 6 tetrahedra per cube arrangements are within 5% of each other, signifying that there is no great difference in
accuracy of calculations using either of the two arrangements. The corresponding 64-node and 125-node Lagrangian cubic elements, which are suitable for incompressibility analysis, have values of $f_n^*$ in the range 10 to 20% less than the tetrahedral elements.

Equation (23) can be generalised for the Lagrangian cube family in terms of the parameter $N$, the order to which the polynomial interpolation displacement function is complete, by substituting in Equations (13) and (14) to give

Lagrangian cube:

$$f_n^* = 3 - \frac{(N + 1)^3 - 1}{N^3}$$

(24)

It is of interest to examine Equation (24) as we progress to very high order elements, that is $N \to \infty$. This gives the result

Lagrangian cube:

$$\lim_{N \to \infty} f_n^* = 2$$

(25)

Thus the free degrees-of-freedom per node will approach two as we progress to very high order Lagrangian cubic elements. Equation (23) can also be generalised for the tetrahedron family by substituting in Equations (15) to (17). In this case it is similarly found that, for both element arrangements considered, the free degrees-of-freedom per node tends to 2 as $N \to \infty$. The interpretation of this result is that each node introduces three degrees-of-freedom, while the number of incompressibility constraints imposed per node tends to one for elements of a very high order. For the two-dimensional case it can similarly be demonstrated the free degrees-of-freedom per node tends to unity, since in this case each node only introduces two degrees-of-freedom.

4. CONSIDERATION OF COMPUTATIONAL EFFORT

A major factor in the application of finite elements in two dimensional and especially three dimensional problems is the consideration of computational effort. The parameter free degrees-of-freedom per node ($f_n^*$) developed in this paper, does not, however, take this into account. So, while progressing to higher order elements provides greater values of $f_n^*$ (Tables 1 to 4), this gain can be offset by a computational effort which increases at a greater rate. Hence a trade-off is required between the number of free degrees-of-freedom introduced per node and the computational effort involved for that node. This trade-off can be quantitatively determined using a further new parameter "free degrees-of-freedom per computational effort" ($g_n^*$) defined as
\[ f_b = \frac{f_n}{b_n} \]  

where \( f_b \) is the free degrees-of-freedom per node defined in Equation (19) and \( b_n \) is a measure of the "computational effort per node".

In choosing a finite element for incompressibility analysis, \( f_b \) is a direct measure of the computational performance of a given finite element. However, it is difficult to generalise \( f_b \) for a particular element type since \( b_n \) is strongly dependent on the type of finite element analysis performed (Cook, 1981). The type of analysis can vary from linear static to non-linear dynamic problems with non-linear analysis involving a wide range of solution techniques (Mondkar and Powell, 1978; Sloan, 1981). The following discussions for two and three dimensional problems is therefore limited to a more qualitative examination of \( f_b \).

The linear strain triangle is thought to be the most computationally efficient element for plane strain incompressibility conditions. That is, the next higher order element, the quadratic strain triangle, has a value of \( f_b \) only 1.33 times greater than the linear strain triangle \( (\frac{2}{3})^{1/2} = 1.33) \). The corresponding ratio of computational effort per node \( (b_n) \) is thought to be considerably greater than 1.33 for most plane strain two dimensional analyses and therefore the linear strain triangle would have a greater value of \( f_b \) (free degrees-of-freedom per computational effort). For a given level of accuracy, it would therefore be more computationally efficient to use the linear strain triangle with a mesh of more nodes than higher order suitable elements.

For axisymmetric conditions, it is thought possible that triangular elements of a higher order than the 15-node cubic strain triangle element (considered in Table 4) may be more computationally efficient, when using the standard finite element displacement method. However, the complexity of coding associated with using such high order elements may make the 15-node triangle an overall better choice.

The consideration of computational effort is most important in three dimensional finite element analysis, where the computational effort can be up to 3 orders of magnitude greater than equivalent two dimensional analyses (Zienkiewicz, 1977; Cook, 1981). It is generally accepted that Lagrangian cube elements are computationally inefficient due to the large number of internal nodes and the incompleteness of the polynomial interpolation functions (Zienkiewicz, 1977), and the Serendipity cube elements are generally used in preference. However, this paper has established that the Serendipity cube is unsuitable for incompressible material analysis. It would therefore appear that the Tetrahedron family of elements is the most computationally efficient for incompressible material analysis. For the three-dimensional finite element model and results
presented in section 5, the parameter describing the free degrees-of-freedom per computational effort \( f_s \) is probably at a maximum somewhere in the range of the 20-node quadratic strain tetrahedron and the 35-node cubic strain tetrahedron. However, it must be emphasised that these considerations are dependent on the type of analysis performed, and for non-linear analysis, the solution scheme adopted.

5. NUMERICAL COMPARISON OF 10-NODE AND 20-NODE TETRAHEDRA

To substantiate some of the findings of the previous sections, this section presents the results of two numerical experiments comparing the performance of the 10-node linear strain tetrahedron and 20-node quadratic strain tetrahedron. For the 6 tetrahedra per cube arrangement adopted in these experiments, the 10-node tetrahedron is on the border-line of suitability \( f_s = 0.0 \) while the 20-node tetrahedron is expected to be suitable \( f_s = 0.78 \). The first experiment involved the analysis of a smooth rigid circular footing, vertically loaded at the surface of a semi-infinite elastic half-space. The exact analytical solution to this set of boundary conditions is given by Poulos and Davis (1974) to be

\[
\mathbf{u}_v = \left[ \frac{1 - \nu}{4GR} \right] \mathbf{V}
\]  

(27)

where \( \mathbf{u}_v \) and \( \mathbf{V} \) are the vertical displacement and load respectively, \( G \) and \( \nu \) are the elastic shear modulus and Poisson's ratio of the half-space, and \( R \) is the radius of the circular footing. To investigate the effects of incompressibility, a series of elastic finite element analyses were performed for \( \nu \) ranging from 0.4 to 0.49999. The errors associated with the linear strain and quadratic strain tetrahedra analyses were calculated in terms of the error in the calculated vertical stiffness \( K \):

\[
\text{Error(\%)} = \frac{K_p - K_e}{K_e} \times 100
\]  

(28)

where

\[
K = \frac{V(1 - \nu)}{4GRu_v}
\]  

(29)

where the subscripts p and e denote predicted and exact quantities, respectively. The results are presented in Figure 5, where the effect of Poisson's ratio \( \nu \) is quantified through the expression \( \log_{10}(2/1 - 2\nu) \). The results show that under compressible conditions (\( \nu = 0.4 \)), the linear strain and quadratic strain tetrahedra meshes give the same result. But as incompressible conditions are approached (\( \nu \rightarrow 0.5 \)), the error associated with the linear strain element increases at a much
greater rate than the quadratic strain element. Furthermore the former is seen to be still increasing at \( v = 0.49999 \) \((\log_{10}(2/(1 - 2v)) = 5)\), while the error associated with the latter has become constant.

In the second experiment, the vertical collapse load for a rigid rough circular footing at the surface of a weightless elastic perfectly-plastic von Mises material was investigated. The non-linear load-displacement curves for the 10-node linear strain and 20-node quadratic strain tetrahedra meshes are presented in Figure 6. No exact collapse load is known for this problem, however, for a Tresca yield criterion, Eason and Shield (1960) calculated for a rough footing, using upper and lower bound techniques, a vertical collapse load of \( 6.05s_v/(\pi R^2) \), where \( s_v \) is the shear (cohesive) strength of the soil. The collapse load for the von Mises material is therefore greater than \( 6.05s_v/(\pi R^2) \) and less than \( 6.05s_v/(\pi R^2) \) multiplied by \( 2/\sqrt{3} \). This is because in a triaxial stress state the von Mises yield criterion is equivalent to the Tresca yield criterion and in plane strain the von Mises criterion represents the Tresca criterion multiplied by a factor of \( 2/\sqrt{3} \). For a circular footing, the stress states are intermediate between triaxial and plane strain.

The 20-node tetrahedra mesh predicts the collapse load to be within the upper and lower limits discussed previously. The 10-node tetrahedra mesh gives a promising result in that a limit load is clearly reached, but this value is well above both the 20-node tetrahedron collapse load and the upper limit for a von Mises material. Analysis of this problem using a two dimensional axisymmetric formulation, 15-node triangles and a very refined mesh, suggests the exact collapse load to be about \( 6.65s_v/(\pi R^2) \) (Bell, 1991). The 20-node tetrahedron collapse load, shown in Figure 6, is therefore within 3% of the exact collapse load. The corresponding error for the 10-node tetrahedron is estimated to be +13%.

The non-linear three dimensional finite element analysis presented in Figure 6 assumed small strains, and the solution scheme adopted is similar to that used by Sloan and Randolph (1982) and Burd and Houlby (1990). The scheme is a variant of the Euler procedure where at the completion of each load step the unbalanced nodal loads are computed and added onto the applied loads for the next step. This counteracts the tendency to drift from equilibrium and is referred to by Sloan (1981) as a "modified Euler procedure". For further details on the finite element model adopted, the reader is referred to Bell (1991).

Two dimensional representations of the 10 and 20 node tetrahedra meshes analysed are included in Figures 5 and 6. The meshes were formed by generating a two dimensional mesh of triangles and rotating it about the vertical axis to form a series of vertical sectors. Each resulting distorted triangular prism was divided into 3 tetrahedra as shown in Figure 4c. In terms of degrees of freedom per element, this arrangement is equivalent to a 6 tetrahedra per cube arrangement. The mesh dimensions differ substantially for the elastic and elasto-plastic analyses of Figures
5 and 6. The proximity of mesh boundaries significantly affects the accuracy of an elastic analysis and therefore very large mesh dimensions were used in this case (Figure 5). In Figure 6 where the estimation of the plastic collapse load is of the main concern, the mesh boundaries need only contain the immediate collapse mechanism (Bell, 1991). For a given experiment, the number of nodes in the 10 and 20 node tetrahedra meshes were within 10% and the total computation times were just as similar. Included in Figures 5 and 6 is the number of unrestrained degrees-of-freedom, which in particular for the 10-node tetrahedron meshes always exceeded the number of incompressibility constraints.

6. DISCUSSION

The results in Figures 5 and 6 support the findings of the previous sections. Under incompressibility conditions for meshes with similar numbers of nodes, the 20-node quadratic strain tetrahedron clearly performs better than the 10-node linear strain tetrahedron. This greater accuracy is a result of more free degrees-of-freedom being available in the 20-node tetrahedron mesh. For example, using the elastic analysis in Figure 5, the 20-node tetrahedron mesh (Mesh ESS20) has a total number of 11,585 unrestrained degrees-of-freedom and 7,920 incompressibility constraints (792 elements x 10 constraints per element) which leaves a remainder of 3,665 free degrees-of-freedom. A similar calculation for the 10-node tetrahedron mesh (Mesh ESS10) gives a much smaller number of 753 free degrees-of-freedom. A similar comparison can be made for the meshes of Figure 6, which have 2,049 and 268 free degrees-of-freedom respectively.

Sloan (1981) performed a large number of vertical collapse load calculations for plane strain and axisymmetric conditions, using various unsuitable, marginal and suitable two-dimensional finite elements. Close examination of his study shows agreement with the collapse load experiment of Figure 6 and the findings of this paper. Table 5 summarises the experiments of Sloan (1981) in terms of free degrees-of-freedom and error in collapse load prediction, and includes the corresponding numbers from Figure 6. For plane strain conditions, the 8-node and 12-node quadrilaterals, which are on the border-line of suitability ($\xi_s = 0$), are observed to produce acceptably accurate results as long as the number of free degrees-of-freedom exceed about 10. Unfortunately this corresponds to a very low ratio of free degrees-of-freedom to unrestrained degrees-of-freedom and is therefore very inefficient. For a given level of accuracy, a better computational performance is provided by an element such as the 15-node cubic strain triangle, which introduces free degrees-of-freedom within each element and therefore has a considerably higher ratio of free degrees-of-freedom to unrestrained degrees-of-freedom. The one anomalous result in Table 5 is for a linear strain triangle, an element which would usually be expected to perform well ($\xi_s = 0.5$), in which a plane strain analysis involving 155 free degrees-of-freedom produced an overshoot of the collapse load by 7%. The explanation may
be in the fact that for this analysis only, Sloan (1981) used a 7-point integration scheme when a three point integration is all that is necessary. This points to the possibility that “over-integration” of the element stiffness may result in further constraints on the solution, although the precise mechanism by which this could occur is unclear.

Table 5 shows similar trends for both two and three dimensional axisymmetric analyses, except for this case at least 50 free degrees-of-freedom (in the two-dimensional sense) are required for the vertical collapse load to be predicted within 5%. The equivalent number of two-dimensional degrees-of-freedom for three-dimensional analysis are obtained by dividing by the number of radial planes of nodes in the mesh. The comparison should be regarded as only approximate. The issue of computational efficiency is once again reiterated in Table 5 where the 10-node linear strain tetrahedron ($f^e_\iota = 0$) is considerably less accurate than the 20-node quadratic strain tetrahedron ($f^e_\iota = 0.78$) for meshes with similar numbers of unrestrained degrees-of-freedom and total computational effort.

The favourable performance of elements on the border-line of suitability ($f^e_\iota = 0$), such as the 10-node linear strain tetrahedron and 8-node and 12-node quadrilateral elements (in plane strain), may be a consequence of more than just free boundary degrees-of-freedom. Figure 5 demonstrates the detrimental effects of incompressibility to be not fully apparent when $\nu=0.49$, with this being the value of Poisson’s ratio adopted in many nominally incompressible analyses (Sloan, 1981; Sloan and Randolph, 1982), including the non-linear analysis presented in Figure 6. This may be one reason why elements with $f^e_\iota = 0$ have given favourable results. A further reason may be a consequence of the mesh topology adopted. Special arrangements of elements can result in the total number of incompressibility constraints being less than the product of the number of elements and the number of constraints per element (Nagtegaal et al, 1974; Burd and Houlsby, 1990).

7. CONCLUSIONS

Taking into consideration the new parameters introduced in this paper, the analysis of Sloan and Randolph (1982), which examined the suitability of the two-dimensional triangular and Serendipity quadrilateral finite elements for plane strain and axisymmetric conditions, was repeated. For axisymmetric incompressibility conditions, the 15-node cubic strain triangle is the lowest order suitable element using the standard finite element displacement method. For plane strain, the linear, quadratic and cubic strain triangles and the 17-node quadrilateral elements were found to be clearly suitable since free degrees-of-freedom are introduced per element. The constant strain triangle and 8-node and 12-node quadrilateral elements have a value of zero free degrees-of-freedom per element (similarly per node). These elements are suitable under plane strain conditions as long as the mesh is sufficiently refined, and the total number of unrestrained
degrees-of-freedom exceed the total number of incompressibility constraints. However, when computational efficiency is considered, that is, free degrees-of-freedom per computational effort, the 8-node and 12-node quadrilaterals are found to be inferior to elements which introduce free degrees-of-freedom within each element and therefore internally within the mesh. The most computationally efficient element for plane strain incompressibility analysis is thought to be the 6-node linear strain triangle.

The parameters examined in this paper for three dimensional finite elements are dependent on the configuration of the elements adopted. Therefore the three families of three dimensional finite elements investigated for accurate incompressibility analysis were taken to be configured in a regular cubic arrangement. The first two elements of the Lagrangian cube family, the 8-node and 27-node cubes, were found to be unsuitable while all higher order elements are suitable. The first three elements of the Serendipity cube family are unsuitable, and it is suspected that higher order elements, if they do exist, are also unsuitable. The examination of the tetrahedron family was divided into two categories since a cube can be divided into either 5 and 6 tetrahedra. The constant strain tetrahedron is unsuitable for both categories. The linear strain tetrahedron is suitable when used in a 5 tetrahedra per cube arrangement and is on the border-line of suitability in a 6 tetrahedra per cube arrangement, since the number of free degrees-of-freedom per element (similarly per node) is zero. The latter is thought to be only suitable if the mesh boundary nodes are not over-constrained, although it must be stressed that the analysis would be very inefficient because of the poor ratio of free to total degrees-of-freedom. All subsequent higher order tetrahedra produce free degrees-of-freedom and are therefore suitable. When computational effort is considered, the quadratic strain and cubic strain tetrahedra are thought to be significantly more efficient than the linear strain tetrahedron and suitable Lagrangian cubic elements.

ACKNOWLEDGEMENT

The first named author gratefully acknowledges the support he receives as a Commonwealth scholar.

REFERENCES


**NOTATION**

- \( b_n \): computational effort per node
- \( c_e \): number of constraints per element
- \( d_c' \): ratio of the degrees-of-freedom to constraints in the limit as the mesh is uniformly refined
- \( d_e' \): number of degrees-of-freedom per element in the limit as the mesh is uniformly refined
- \( e \): number of elements
- \( f_e \): number of free degrees-of-freedom per element in the limit as the mesh is uniformly refined
- \( f_n \): number of free degrees-of-freedom per node in the limit as the mesh is uniformly refined
- \( G \): elastic shear modulus
- \( k \): number of mesh interior nodes
- \( K \): non-dimensionalised vertical stiffness
- \( l \): number of local nodes per element
\( m \) is the sum of the internal nodal angles divided by \( \pi \)

\( n \) number of nodes

\( n_e^* \) ratio of number of nodes to number of elements in the limit as the mesh is uniformly refined

\( N \) the order to which the element polynomial interpolation displacement function is complete

\( r \) number of mesh edge nodes

\( R \) radius of a circular footing

\( s \) number of mesh face nodes

\( s_u \) shear (cohesive) strength

\( \hat{u}, \hat{v}, \text{ and } \hat{w} \) velocities in the x, y and z directions

\( u_V \) vertical footing displacement

\( V \) vertical footing load

\( \dot{\varepsilon}_{kk} \) volumetric strain rate

\( v \) Poisson's ratio

\( \theta \) internal nodal angle
Table 1. Suitability of Lagrangian cubic elements for incompressible material analysis

<table>
<thead>
<tr>
<th>Element type</th>
<th>N²</th>
<th>Degrees-of-freedom per element ($d_e^*$)</th>
<th>Constraints per element ($c_e$)</th>
<th>Degrees-of-freedom per constraint ($d_e^* = d_e^*/c_e$)</th>
<th>Suitable</th>
<th>Free degrees-of-freedom per element ($J_e^*$)</th>
<th>Free degrees-of-freedom per node ($J_e^*$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8-node cube</td>
<td>1</td>
<td>3</td>
<td>7</td>
<td>0.43</td>
<td>No</td>
<td>-4</td>
<td>-4.00</td>
</tr>
<tr>
<td>27-node cube</td>
<td>2</td>
<td>24</td>
<td>26</td>
<td>0.92</td>
<td>No</td>
<td>-2</td>
<td>-0.25</td>
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<tr>
<td>64-node cube</td>
<td>3</td>
<td>81</td>
<td>63</td>
<td>1.29</td>
<td>Yes</td>
<td>18</td>
<td>0.67</td>
</tr>
<tr>
<td>125-node cube</td>
<td>4</td>
<td>192</td>
<td>124</td>
<td>1.55</td>
<td>Yes</td>
<td>68</td>
<td>1.06</td>
</tr>
</tbody>
</table>

Notes:
1. Refer to Figure 1. for schematic diagrams of Lagrangian cubic elements.
2. Order of "complete" polynomial displacement function.
3. Based on assumption of exact integration of a straight-sided element.
4. * represents the limiting case of a uniformly refined mesh.

Table 2. Suitability of Serendipity cubic elements for incompressible material analysis

<table>
<thead>
<tr>
<th>Element type</th>
<th>N²</th>
<th>Degrees-of-freedom per element ($d_e^*$)</th>
<th>Constraints per element ($c_e$)</th>
<th>Degrees-of-freedom per constraint ($d_e^* = d_e^*/c_e$)</th>
<th>Suitable</th>
<th>Free degrees-of-freedom per element ($J_e^*$)</th>
<th>Free degrees-of-freedom per node ($J_e^*$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8-node cube</td>
<td>1</td>
<td>3</td>
<td>7</td>
<td>0.43</td>
<td>No</td>
<td>-4</td>
<td>-4.00</td>
</tr>
<tr>
<td>20-node cube</td>
<td>2</td>
<td>17</td>
<td>12</td>
<td>0.71</td>
<td>No</td>
<td>-5</td>
<td>-1.25</td>
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<tr>
<td>32-node cube</td>
<td>3</td>
<td>29</td>
<td>21</td>
<td>0.72</td>
<td>No</td>
<td>-8</td>
<td>-1.14</td>
</tr>
</tbody>
</table>

Notes:
1. Refer to Figure 2. for schematic diagrams of Serendipity cubic elements.
2. Order of "complete" polynomial displacement function.
3. Based on assumption of exact integration of a straight-sided element.
4. * represents the limiting case of a uniformly refined mesh.
Table 3. Suitability of tetrahedral elements for incompressible material analysis

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
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<tbody>
<tr>
<td>Constant strain</td>
<td>1</td>
<td>1</td>
<td>0.6</td>
<td>0.60</td>
<td>No</td>
<td>-0.4</td>
<td>-2.00</td>
<td>0.5</td>
<td>0.5</td>
<td>No</td>
<td>-0.5</td>
<td>-3.0</td>
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<tr>
<td>Linear strain</td>
<td>2</td>
<td>4</td>
<td>4.2</td>
<td>1.05</td>
<td>Yes</td>
<td>0.2</td>
<td>0.14</td>
<td>4.0</td>
<td>0.0</td>
<td>#</td>
<td>0.0</td>
<td>0.0</td>
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<tr>
<td>Quadratic strain</td>
<td>3</td>
<td>10</td>
<td>13.8</td>
<td>1.38</td>
<td>Yes</td>
<td>3.8</td>
<td>0.83</td>
<td>13.5</td>
<td>3.5</td>
<td>Yes</td>
<td>3.5</td>
<td>0.78</td>
</tr>
<tr>
<td>Cubic strain</td>
<td>4</td>
<td>20</td>
<td>32.4</td>
<td>1.62</td>
<td>Yes</td>
<td>12.4</td>
<td>1.15</td>
<td>32.0</td>
<td>12.0</td>
<td>Yes</td>
<td>12.0</td>
<td>1.12</td>
</tr>
</tbody>
</table>

Notes:
1. Refer to Figure 3 for schematic diagrams of tetrahedral elements.
2. Order of "complete" polynomial displacement function.
3. Based on assumption of exact integration of a straight-sided element.
4. # on the border-line of suitability. Considered suitable for a refined mesh with the number of unrestrained degrees-of-freedom exceeding the number of incompressibility constraints.
5. * represents the limiting case of a uniformly refined mesh.
Table 4. Suitability of triangular and Serendipity quadrilateral elements for incompressible material analysis

<table>
<thead>
<tr>
<th>Element type</th>
<th>Plane strain</th>
<th>Axisymmetry</th>
<th>Suitable</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Degrees-of-freedom per element ($d^*_e$)</td>
<td>Constraints$^{1,2}$ per element ($c^*_e$)</td>
<td>Degrees-of-freedom per constraint ($d^<em>_e = d^</em>_e/c^*_e$)</td>
</tr>
<tr>
<td>Constant strain triangle</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Linear strain triangle</td>
<td>4</td>
<td>3</td>
<td>$\frac{4}{3}$</td>
</tr>
<tr>
<td>Quadratic strain triangle</td>
<td>9</td>
<td>6</td>
<td>$\frac{2}{2}$</td>
</tr>
<tr>
<td>Cubic strain triangle</td>
<td>16</td>
<td>10</td>
<td>$\frac{8}{7}$</td>
</tr>
<tr>
<td>4-node quadrilateral</td>
<td>2</td>
<td>3</td>
<td>$\frac{2}{3}$</td>
</tr>
<tr>
<td>8-node quadrilateral</td>
<td>6</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>12-node quadrilateral</td>
<td>10</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>17-node quadrilateral</td>
<td>16</td>
<td>14</td>
<td>$\frac{8}{7}$</td>
</tr>
</tbody>
</table>

Notes:
1. Data after Sloan and Randolph (1982).
2. Based on assumption of exact integration of a straight-sided element.
3. # on the border-line of suitability. Considered suitable for a refined mesh with the number of unrestrained degrees-of-freedom exceeding the number of incompressibility constraints.
4. * represents the limiting case of a uniformly refined mesh.
Table 5. The relationship between free degrees-of-freedom and collapse load prediction for two and three dimensional finite elements

<table>
<thead>
<tr>
<th>Element Type</th>
<th>Free degrees-of freedom per node (京津)</th>
<th>Mesh Reference</th>
<th>Unrestrained degrees-of freedom</th>
<th>Incompressibility constraints</th>
<th>Free degrees-of freedom</th>
<th>Error in collapse load (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PLANE STRAIN</strong></td>
<td></td>
<td></td>
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<tr>
<td>8-node quadrilateral</td>
<td>0</td>
<td>SRF8</td>
<td>156</td>
<td>150</td>
<td>6</td>
<td>13</td>
</tr>
<tr>
<td>8-node quadrilateral</td>
<td>0</td>
<td>SRF9</td>
<td>392</td>
<td>384</td>
<td>8</td>
<td>10</td>
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<tr>
<td>8-node quadrilateral</td>
<td>0</td>
<td>SRF4</td>
<td>496</td>
<td>486</td>
<td>10</td>
<td>5</td>
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<tr>
<td>8-node quadrilateral</td>
<td>0</td>
<td>SRF10</td>
<td>726</td>
<td>714</td>
<td>12</td>
<td>4</td>
</tr>
<tr>
<td>12-node quadrilateral</td>
<td>0</td>
<td>SRF1</td>
<td>94</td>
<td>90</td>
<td>4</td>
<td>#</td>
</tr>
<tr>
<td>12-node quadrilateral</td>
<td>0</td>
<td>SRF2</td>
<td>257</td>
<td>250</td>
<td>7</td>
<td>12</td>
</tr>
<tr>
<td>12-node quadrilateral</td>
<td>0</td>
<td>SRF5</td>
<td>503</td>
<td>490</td>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td>12-node quadrilateral</td>
<td>0</td>
<td>SRF3</td>
<td>739</td>
<td>720</td>
<td>19</td>
<td>4</td>
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<tr>
<td>Linear strain triangle</td>
<td>0.5</td>
<td>SRF6</td>
<td>584</td>
<td>429 (^2)</td>
<td>155</td>
<td>7</td>
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<tr>
<td>Cubic strain triangle</td>
<td>0.75</td>
<td>SRF7</td>
<td>786</td>
<td>480</td>
<td>306</td>
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<tr>
<td><strong>TWO DIMENSIONAL AXISYMMETRY</strong></td>
<td></td>
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<tr>
<td>12-node quadrilateral</td>
<td>-0.6</td>
<td>SRF3</td>
<td>726</td>
<td>936</td>
<td>-210</td>
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<tr>
<td>Linear strain triangle</td>
<td>-1.0</td>
<td>SRF6</td>
<td>584</td>
<td>858</td>
<td>-274</td>
<td>#</td>
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<tr>
<td>Cubic strain triangle</td>
<td>0.125</td>
<td>SRF7</td>
<td>786</td>
<td>720</td>
<td>66</td>
<td>5</td>
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<tr>
<td><strong>THREE DIMENSIONAL AXISYMMETRY</strong></td>
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<tr>
<td>Linear strain tetrahedron</td>
<td>0</td>
<td>PSR10</td>
<td>7084</td>
<td>6816</td>
<td>268 (16)(^4)</td>
<td>13</td>
</tr>
<tr>
<td>Quadratic strain tetrahedron</td>
<td>0.78</td>
<td>PSR20</td>
<td>7029</td>
<td>4980</td>
<td>2049 (108)(^4)</td>
<td>3</td>
</tr>
</tbody>
</table>

Notes:
1. SRF* mesh results taken from Sloan (1981).
2. * load not observed to achieve a clearly defined limit value.
3. Using a 7-point integration scheme.
4. Numbers in brackets correspond to an equivalent two-dimensional mesh.
FIGURE CAPTIONS

Figure 1: Lagrangian cube family: (a) 8-node cube, (b) 27-node cube, (c) 64-node cube*, (d) 125-node cube*; (* - only visible nodes on immediate faces are shown).

Figure 2: Serendipity cube family: (a) 8-node cube, (b) 20-node cube, (c) 32-node cube.

Figure 3: Tetrahedron family: (a) 4-node constant strain tetrahedron, (b) 10-node linear strain tetrahedron, (c) 20-node quadratic strain tetrahedron, (d) 35-node cubic strain tetrahedron*; (* - only face nodes on shaded face are shown).

Figure 4: Tetrahedra arrangements: (a) 5 tetrahedra per cube, (b) 6 tetrahedra per cube, (c) 3 tetrahedra per triangular prism.

Figure 5: The comparison of 10-node and 20-node tetrahedra for a vertically loaded smooth rigid circular surface footing under elastic incompressibility conditions.

Figure 6: The comparison of 10-node and 20-node tetrahedra for the prediction of the vertical collapse load of a rigid rough circular footing at the surface of a von Mises material.
Mesh ESS10

- 3,951 nodes
- 2,464 10-node linear tetrahedra
- 10,609 unrestrained d.o.f.

Mesh ESS20

- 4,312 nodes
- 792 20-node quadratic tetrahedra
- 11,585 unrestrained d.o.f.
The graph shows the relationship between $V / \pi R^2 s_n$ and $\mu V / R$ for two different meshes, labeled as Mesh PSR10 and Mesh PSR20.

The vertical axis represents the dimensionless shear force $V / \pi R^2 s_n$, and the horizontal axis represents the ratio $\mu V / R$. Two curves are plotted, one for each mesh, distinguishing them by solid and dashed lines.

The equation $G/s_n = 100$, $\nu = 0.49$ is used to normalize the shear force.

**Plan View and Vertical Section**

- **Mesh PSR10**
  - 2,745 nodes
  - 1,704 10-node linear tetrahedra
  - 7,084 unrestrained d.o.f.

- **Mesh PSR20**
  - 2,713 nodes
  - 498 20-node quadratic tetrahedra
  - 7,029 unrestrained d.o.f.