"RESEARCH AT OXFORD ON ANALYSIS OF THE PIEZOCONE TEST, 1988"

by

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This report contains six papers on the analysis of the piezocone test published in 1988:

Research on the Analysis of the Piezocone at Oxford University.
   G.T. Houlsby and C.P. Wroth

Analysis of the Cone Penetration Test by the Strain Path Method.
   C.I. Teh and G.T. Houlsby

Analysis of the Piezocone in Clay.
   G.T. Houlsby and C.I. Teh

Undrained deep penetration, I: shear stresses.
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Elastic-plastic incompressible flow around an infinite cone.
   C. Sagaseta and G.T. Houlsby

The Piezocone Penetration Test.
   G.T. Houlsby
Research on the Analysis of the Piezocone at Oxford University

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Introduction

Since 1983 a programme of research on the analysis of the piezocone has been in progress at Oxford University. This research is in parallel with experimental work on the piezocone both in large calibration chambers and in the field, directed by Dr. G.C. Sills, but only the theoretical developments are described here. The research has focused on the prediction of stresses and pore pressures on a cone penetrating into a homogeneous bed of clay under undrained conditions, and on the subsequent dissipation of pore pressures. Only after these problems have been solved will the more complex problems of penetration under drained conditions and into non-uniform materials be studied. Even these simplified problems pose severe difficulties for any analysis.

Predictions of cone factors by approximate theories were based previously either on modifications of bearing capacity theory, or on the concepts of cavity expansion. Whilst the former invariably do not obey the correct boundary conditions, the latter do not reproduce correctly the stress path followed by soil elements.

This research has been based on the observation by Baligh (1986) that problems of undrained deep cone penetration into uniform materials can be represented as steady flow problems of soil past the cone. He suggested the use of the strain path method, in which an approximate flow field around the cone is used to estimate strains and hence stresses. Since the equilibrium equations are not satisfied the stresses must be approximate only; but
because the problem is very heavily constrained kinematically the flow field cannot be significantly in error. Much of the research programme has focussed on ways by which an appropriate correction for equilibrium can be made.

The choice of this approach has not been arbitrary, but is based on a preliminary study of the problem (Houlsby et al., 1985) in which the governing equations were studied in detail. The equations were found to be of mixed hyperbolic and elliptical character in both the elastic and plastic regions, with the streamlines forming the characteristics of the hyperbolic equations. Methods based on integration along streamlines therefore have a sound mathematical basis. It is more difficult to determine how the elliptical nature of the problem, which causes transfer of information across streamlines, can be properly accounted for in any solution.

The research has been divided into two principal areas of development:

(i) closed form solutions
(ii) numerical methods.

In both cases only a very simple model of soil behaviour has been used - a linear elastic-perfectly plastic model with a Von Mises yield surface.

**Closed form solutions**

The development of appropriate closed form solutions is for two purposes. Firstly they allow solutions to be found to simplified problems which either can be compared with the numerical solutions for realistic penetration problems, or provide useful bounding solutions for extreme values of parameters. The use of cylindrical cavity expansion theory has proved extremely valuable in this context. Secondly closed form solutions can be used within numerical analyses to provide detail which cannot be found by conventional numerical methods; an example is the solution for the singularities of stress which occur at the cone tip.
Cylindrical cavity expansion theory

Cylindrical cavity expansion theory has been used in the past to provide estimates of the radial stresses on the shaft of a cone, which may be approximated by the limit pressure for cavity expansion. In order to apply these ideas to cone analysis two significant modifications have been made to the theory. Firstly, in order that the applications are not limited to soils for which \( K_o = 1.0 \), solutions have been developed which account for anisotropic as well as isotropic initial stresses. Secondly the theory has been generalised to take account of an axial shear stress on the surface of the cavity, and is thus able to account more realistically for penetration of cones on slightly rough shafts.

In addition to these developments the rigour of the analysis has been examined, with solutions being developed which account for large strains in both the elastic and plastic regions. The more conventional approach (e.g. by Gibson and Anderson, 1961) accounts for large strain in the plastic region but not in the elastic region. The effects of including the Jaumann terms in the definition of the stress rate (for the case where there is an axial shear stress present) have also been examined.

Cavity expansion theory has been shown formally to yield the correct solution to the problem of penetration of a cone of small included angle (e.g. less than approximately 10°). Whilst not of practical importance for the CPT, this result does indicate that penetration devices with small taper angles could yield results which would be relatively easily interpreted.

The stress distributions from cavity expansion theory have been compared with the numerical results for a 60° cone, and indicate that there is in fact some unloading just behind the cone shoulder. Finally the results of the cavity expansion solution have been used in the choice of appropriate outer boundary conditions for the numerical analysis.
Cone tip singularity

The other major application of closed form solution has been to examine the stress conditions, very close to the tip of the cone (Sagaseta and Houlsby, 1988). This region is difficult to deal with in any numerical solution, but if viewed on a sufficiently small scale appears as a classical singularity problem. The penetration of a conical surface extending to infinity into an undrained clay is considered.

The problem has been solved by standard methods of separation of variables using a polar coordinate system. It is found that the solutions for strain rate and velocity must be power functions of the radial coordinate, that deviator stresses are constant with radius and that a logarithmic singularity in the mean stress must occur. (In practice of course such a singularity would be removed by the slight compressibility of the soil).

Two types of solution are found. In the first there is an inner region adjacent to the cone of plastically deforming soil, with outside it the soil remaining elastic. The elastic-plastic boundary is also a conical surface. This type of solution is found only to apply to relatively sharp cones (included angles less than 20°, but depending slightly on the cone roughness). At very small cone angles the results of the singularity solution become identical to the results in which the penetration of the cone is considered as a cylindrical cavity expansion.

For larger cone angles no solution with elastic and plastic regions exists, and all the soil around the tip of the cone must be deforming plastically. This solution applies for all realistic cone geometries, and is interpreted as indicating that the boundary between elastic and plastic material must be ahead of the cone tip and not passing through it.

Although included under the heading of closed form solutions, the cone tip singularity results cannot be expressed analytically and require the solution
of simultaneous ordinary differential equations, which has been achieved using
shooting methods.

Numerical analysis

The numerical analysis programme has concentrated on development of the
strain path method, emphasising particularly methods of assessing the
importance of the inequilibrium problem and of achieving an appropriate
correction to the velocity field in order to eliminate the inequilibrium (Teh,
1987, and Teh and Houlsby, 1988).

The magnitude of the out of balance forces due to the lack of equilibrium
in the strain path solution can be reduced by making the best possible initial
estimate of the flow field around the cone. This has been achieved by
obtaining the solution to the equivalent fluid flow problem, using either
inviscid or viscous theory. Careful attention was paid to the numerical
methods necessary to obtain an accurate solution, which was achieved by a
Successive-Over-Relaxation method and the use of Second Upwind Differencing to
calculate the correct vorticity transport terms.

From the appropriate velocity solution from fluid mechanics, the
streamline positions and the strain rates were calculated. The constitutive
equations for the deviatoric stresses were integrated up the streamlines from
known starting conditions to obtain a deviatoric stress solution. Estimates
of the mean stress were then made by integrating the appropriate equilibrium
equation in the radial direction from the outer boundary, or axially from the
upstream side. The former was found to be more satisfactory. Because the
flow field is not exactly correct, both equilibrium equations cannot be
satisfied simultaneously.

Three methods of achieving a correction to the flow field have been
attempted, none with complete success. The first is a Newton-Raphson
correction method in which perturbations are applied to the stream function at
each grid point in turn, and the variation of the inequilibrium at every other point determined. The process is computationally very timetaking, but in principle allows the sensitivity to the changes at every point to be determined, and by inversion of a matrix the correction to the flow field may be found. The method is strictly only applicable to linear problems, and it has been found that it is capable of eliminating the errors in a flow involving purely elastic deformation, but is totally unsatisfactory for all realistic flows which involve plastic deformation.

The second approach has been to treat the flow as unsteady: the lack of equilibrium can be used to compute the accelerations of every point if an appropriate density is assumed. This method is less expensive computationally than the first, and has been found to reduce major errors in the flow field, but does not totally eliminate the errors.

The third approach is based on finite element analysis. The results of the strain path analysis are passed to a finite element program, the appropriate out of balance forces which correspond to the inequilibrium are determined and then negated in a conventional finite element analysis. The corresponding displacements are then interpreted as movements of the streamlines. The modified streamline pattern is then used as the starting point of a new strain path analysis. Whilst not justified by any rigorous theoretical understanding, this approach was thought to be reasonable. In fact the method did not work well, but this is thought to be at least in part due to difficulties in interpolation between the rectangular grid used for the flow field computation and the Gauss points in the finite element mesh. This problem exposed the extreme sensitivity of stress distribution to very small alterations in the flow field.

Finally a rather different approach to the problem has been made, combining the merits of the strain path method and finite element analysis. This method has been successful. The strain path method has the great
advantage that it correctly models the steady state nature of the problem, but suffers from the drawback that the solution does not obey the equilibrium equations. On the other hand, conventional incremental finite element analysis obeys the equilibrium equation, but is incapable of modelling the steady state nature of the problem. This is because it is unrealistic to carry out analyses to the extremely high deformations which are necessary to develop the high lateral stresses on the cone shaft which occur in steady state flow. (It is essential to draw a careful distinction between a genuine steady state flow solution and the solution to an incremental displacement collapse problem for a cone initially in a "pre-bored" hole). The problem arises because the initial stresses for the finite element analysis are usually taken as the in-situ stresses. The solution has been obtained by first obtaining a strain path solution, and then using the resulting stresses, which are approximately correct, as the starting point for a finite element analysis. The inequilibrium is first eliminated, and then the cone pushed further into the clay until a steady solution is achieved in the finite element analysis.

Figure 1 shows the results of such analyses. Curve AB shows the load-deflection response of a penetrometer in a "pre-bored" hole, analysed by conventional large displacement finite elements and starting from the in situ stresses. Curve CD shows the result of the calculation as described in the previous paragraph, starting from the stress state given by the strain path analysis and making the appropriate equilibrium correction before analysing the further penetration of the cone. Curve EF shows a similar analysis but without the equilibrium correction. The solution from curve CD is thought to be most realistic, and yields higher cone factors than either the uncorrected strain path analysis, or the conventional finite element analysis.

The cone factor $N_{kt} = (q_t - \sigma_0)/s_u$ is found to depend on the rigidity index $I_r = G/s_u$ the horizontal stress (expressed conveniently in the form of
Figure 1 - Load deflection curves from finite element analysis of penetrometer
with different initial stress conditions

\[ \Delta = \frac{(\sigma_{vo} - \sigma_{ho})}{2s_u} \] and the roughness of both the cone and the shaft.
(Details of the effects of cone roughness could not be achieved by the
analysis described above, but have been estimated by other analytical
methods). It has been found that the results of the analyses can be expressed
as:

\[ N_{kt} = N_s(1.25 + \frac{I_r}{2000}) + 2.4\alpha_f - 0.2\alpha_s - 1.8\Delta \]

where \( \alpha_f \) and \( \alpha_s \) are the roughness factors on the cone face and shaft
respectively, and \( N_s \) is the normalised limit pressure for spherical expansion:

\[ N_s = \frac{\Delta}{3}(1 + \ln(I_r)) \]
For further details of the development of these expressions see Houlsby and Teh (1988). The results of the analysis are shown in Figure 2, which shows the calculated variation of $N_{kt}$ with $I_r$ for different values of $\Delta$. Reasonable values of $\alpha_f$ and $\alpha_d$ have been used. The figure shows clearly that the cone factor cannot be treated as constant and highlights the importance of both stiffness and lateral stress. Both $I_r$ and $\Delta$ will vary with overconsolidation ratio, and to some extent these effects may be cancelling, since both $I_r$ and $\Delta$ will usually decrease with overconsolidation ratio (Wroth and Houlsby, 1985). This will result in perhaps a less dramatic variation of $N_{kt}$ than is at first implied by Figure 2.

Consolidation Analysis

In a second stage of the numerical analysis, an uncoupled consolidation analysis has been carried out, solving the conventional Terzaghi equations. Initial pore pressures were estimated from the stress solutions described above. The principal finding from this study has been the important influence of the rigidity index $I_r$ on the consolidation behaviour. This arises because the region in which there are large initial pore pressures is the region in which plastic deformation occurs, and the size of this region depends on $I_r$. New methods of interpretation of consolidation data to account for this have been proposed (Houlsby and Teh, 1988).

Future developments

In the longer term it is of course intended to carry out fully coupled consolidation analysis, but this depends on the use of a realistic model for the effective stress behaviour of clay during consolidation (e.g. the Modified Cam Clay model). In order to be consistent the penetration phase must also be modelled using the same constitutive law, and this will be one of the next developments for the analysis.
Figure 2 - Variation of calculated cone factor with rigidity index and horizontal stress index

Further work will also be carried out to find a robust algorithm for carrying out the equilibrium correction on the strain path analysis.

Only after these developments are complete will the more difficult task of modifying the analysis for drained steady state penetration into frictional materials be attempted.
This research programme is closely linked to research on the cone pressuremeter, with the use of cavity expansion analysis being common to both. Current developments are being directed mainly towards this area of the analysis.

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References


Analysis of the Cone Penetration Test by the Strain Path Method

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ABSTRACT: A method of analysis for the penetration of a cone penetrometer into clay is presented. The analysis is based on the Strain Path Method, with the initial estimate of the velocity field provided by the solution to the equations of fluid flow. Corrections to account for the equilibrium equations are achieved by the use of large strain finite element analysis. The calculations demonstrate the primary importance of horizontal stress and rigidity index in controlling cone resistance.

Keywords: analysis, clays, cone factor, finite elements, piezocone, strain path

1 INTRODUCTION

The purpose of this paper is to present in outline the method of analysis used to study the penetration of a piezocone into clay. Whilst some of the results are presented here, the intention is to concentrate on the analytical techniques. Full details of the analysis are given by Teh (1987) and a discussion of the results and their implications for the interpretation of piezocone data given by Houlsby and Teh (1988).

The full analysis consists of two stages. In the first the undrained penetration of a CPT into clay is analysed, with the clay idealised as an incompressible elastic-perfectly plastic (Von Mises) material. Values of the cone factor $N_{kt}$ (defined as $(q_t - \sigma_0)/(\sigma_0)$) are derived, as well as stresses and displacements around the cone. In the second stage the consolidation around a static cone is analysed using uncoupled Terzaghi-Readich consolidation theory. Since this second stage involves a straightforward solution to the diffusion equations in axial symmetry, and has been solved using conventional Alternating Direction Implicit techniques, only the first stage of the analysis is described in this paper. A more sophisticated coupled analysis of the consolidation phase is planned as a future research project.

2 METHODS OF ANALYSIS FOR CONE PENETRATION

Approximate derivations of $N_{kt}$ factors have in the past been based mainly on either bearing capacity theory or cavity expansion theory. Modification of the results of plane strain bearing capacity calculations for application to axially symmetric problems is often carried out empirically, although numerical solutions to axisymmetric problems can be obtained using plasticity theory (e.g. Cox et al. (1961), Houlsby and Wroth (1983)). Adaptation of bearing capacity theory to deep penetration problems is much more difficult, and either involves adoption of boundary conditions which are not appropriate for real cone penetrometers (e.g. Koumoto and Kaku (1982)) or results in $N_{kt}$ factors which increase indefinitely with increased penetration (Houlsby and Wroth, 1982). In practice empirical depth factors are applied to shallow bearing capacity solutions.

The problem of the transition between shallow and deep penetration is readily explained by the fact that shallow penetration involves a mechanism in which displaced material moves entirely outwards to the free surface, whilst in deep penetration the displaced material is accommodated by elastic deformation of the soil. Thus cavity expansion theories are applied to deep penetration problems (Vesic, 1972). It is widely held that the
limit solution for the cylindrical cavity expansion pressure \( \psi_c \) is applicable to estimation of the radial stress on the shaft of a penetrometer. For the von Mises yield criterion:

\[
(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 + 6\tau_{rz}^2 = 8\sigma_u^2 \quad \ldots (1)
\]

the solution for cylindrical cavity expansion is:

\[
\psi_c = \sigma_{ro} + \frac{2s_u}{\sqrt{3}(1 + \ln \left[\frac{\sqrt{3}L}{2r}\right])} \quad \ldots (2)
\]

where \( L \) is the rigidity index \( G/s_u \).

Note that \( s_u \) is the undrained strength in triaxial compression. The von Mises criterion implies a shear strength in plane strain of 2\( s_u/\sqrt{3} \).

The spherical cavity expansion pressure \( \psi_s \) is often considered as an estimate of the pressure at the cone tip:

\[
\psi_s = \sigma_{ro} + \frac{4s_u}{3}(1 + \ln(\vert r \vert)) \quad \ldots (3)
\]

Note that both of these expressions demonstrate the important influence of the rigidity index through the terms in \( \ln(\vert r \vert) \).

Baligh (1986) has pointed out that an inconsistency in the application of cavity expansion theory to cone penetration problems is that it does not model correctly the strain paths followed by soil elements. He suggests the use of the Strain Path Method for problems of deep penetration, and this method has been used in this research.

By changing the frame of reference, the penetration of a cone in a homogeneous material can be viewed as a steady state flow of soil past a static penetrometer. An initial estimate of the streamlines must be made, for instance by using fluid mechanics theory. The flow pattern will of course be slightly in error since an incorrect constitutive law has been used, but because the problem is heavily constrained kinematically (by the incompressibility condition and the boundary conditions on the cone) the estimate represents a reasonable first approximation. From this flow pattern the complete history of strain for each soil element may be determined. Using appropriate initial stresses (which may be anisotropic) far ahead of the cone, the deviatoric stresses may be determined by integration of the appropriate constitutive laws up the streamlines. In this study a simple linear elastic-perfectly plastic model with a von Mises yield surface was used, involving only two material parameters, the shear modulus \( G \) and the shear strength in triaxial compression \( s_u \). The appropriate incremental constitutive equations for the steady flow problem (Houlshby et al., 1985) are given by:

\[
\dot{\varepsilon}_{rr} = \frac{\partial u}{\partial r} + \frac{1}{2G} \ddot{\sigma}_{rr} + 6\lambda \dot{\sigma}_{rr} \quad \ldots (4)
\]

\[
\dot{\varepsilon}_{zz} = \frac{\partial v}{\partial z} + \frac{1}{2G} \ddot{\sigma}_{zz} + 6\lambda \dot{\sigma}_{zz} \quad \ldots (5)
\]

\[
\dot{\varepsilon}_{\theta\theta} = \frac{u}{r} = \frac{1}{2G} \ddot{\sigma}_{\theta\theta} + 6\lambda \dot{\sigma}_{\theta\theta} \quad \ldots (6)
\]

\[
\dot{\gamma}_{rz} = \frac{\partial v}{\partial r} + \frac{\partial u}{\partial z} = \frac{1}{G} \ddot{\tau}_{rz} + 12\lambda \dot{\tau}_{rz} \quad \ldots (7)
\]

Where \( u \) and \( v \) are the velocities in the \( r \) (radial) and \( z \) (axial) directions respectively and \( \lambda \) is the plastic multiplier. A superscripted dot indicates the time differential and a dash indicates the deviatoric component. Note of course that the velocities must be such that the incompressibility condition is satisfied. This is most conveniently achieved by obtaining the velocities by the differentiation of a potential function:

\[
u = \frac{1}{r} \frac{\partial \psi}{\partial r}, \quad \dot{\psi} = \frac{1}{r} \frac{\partial \psi}{\partial r} \quad \ldots (8)
\]

The strain rates in steady flow are given by convective equations:

\[
\dot{\sigma}_{rr} = \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rr}}{\partial z} \quad \ldots (9)
\]

and three similar equations for \( \dot{\sigma}_{zz} \), \( \dot{\sigma}_{\theta\theta} \) and \( \dot{\gamma}_{rz} \).

The above equations are supplemented by either \( \lambda = 0 \) for elastic behaviour if the stresses are within the yield surface or equation (1) in the case of plasticity. In addition the equilibrium equations:

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \quad \ldots (10)
\]

\[
\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{rz}}{r} = 0 \quad \ldots (11)
\]

must be satisfied.
Alternative models for the deviatoric stress-strain behaviour of the soil could readily be included in the method. Note that for an incompressible material the mean stress is not determined by the constitutive relations but appears as a reaction which is determined solely by the equilibrium equations. Thus, having calculated the deviatoric stresses, the mean normal stress may be determined by using one of the equilibrium equations, radial or axial, and integration from the outer boundary (which must be set at some point sufficiently distant from the cone). It is then found that the stresses do not exactly obey the other equilibrium relationship, with the discrepancy reflecting the error in the initial flow field. In principle the other relationship could be used to correct the flow field, and the following schemes have been adopted to attempt this correction.

3 NEWTON-RAPHSON CORRECTION

By cross differentiating the equilibrium equations they may be combined to eliminate the mean stress:

\[
\frac{\partial}{\partial z} \left[ \frac{3\sigma''_{rr}}{\partial r} + \frac{3\sigma''_{rz}}{\partial r} + \frac{\sigma''_{rr} - \sigma''_{\theta\theta}}{r} \right] + \frac{\partial}{\partial r} \left[ 3\sigma''_{rz} \frac{\partial z}{\partial r} + 3\sigma''_{zz} \frac{\partial z}{\partial z} + \frac{\sigma''_{rz}}{r} \right] = H
\]

where \( H = 0 \) for a set of stresses in equilibrium, and \( H \) is taken as a measure of the error for a set of stresses which are not in equilibrium.

An initial set of values of the stream function \( \psi_0 \) is estimated and the corresponding values \( \psi \) determined. At each grid point \( \psi \) is then perturbed by a small amount, and the resulting effect on \( H \) determined at every other grid point. The changes are confined to a region downstream and extending a small distance to either side of the perturbed point. After perturbing every grid point in turn an \( NxN \) matrix (where \( N \) is the number of grid points) of "partial derivatives" of \( H \) with respect to \( \psi \) can be calculated. By multiplying \( -H_0 \) by the inverse of this matrix the appropriate correction to \( \psi \) may be determined to eliminate the error approximately (exactly in the case of a linear problem). This method is then repeated with the new \( \psi \) values until the error is reduced to a satisfactory value.

Testing of this approach with some simple problems showed that as long as the perturbed flow does not include any regions in which plastic deformation occurs, then the method converges very rapidly on the correct solution. This is because the equations are purely linear in these cases. If, however, there is any plastic deformation, as is the case for all realistic cone penetration problems, then the method does not converge (even when acceleration or deceleration techniques are also adopted).

4 PSEUDO-DYNAMIC CORRECTION

The second correction method is based on an unsteady flow approach. In this case the equilibrium equations are considered for a material with density \( \rho \):

\[
\frac{\partial \sigma_{rr}}{\partial t} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \sigma_{\theta\theta}}{r} = \rho \left[ \frac{\partial u}{\partial t} + \frac{u}{r} \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial z} \right] \quad \text{...(13)}
\]

and a similar equation in the vertical direction.

Introducing the vorticity \( \xi = \frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} \) and cross differentiating the modified equilibrium equations again to eliminate the mean stress leads to the equation:

\[
\frac{\partial \xi}{\partial t} = \frac{H}{\rho} - \left[ \frac{u}{r} \frac{\partial \xi}{\partial r} + \frac{v}{r} \frac{\partial \xi}{\partial z} - \frac{u}{r} \right] \quad \text{...(14)}
\]

which is a form of the vorticity transport equation.

A choice of a sufficiently small value of \( \rho \) can always be made so that the bracketed term in the above equation is small relative to \( \xi / \rho \). Considering a small time step the value of \( H \) may be used to calculate an increment of \( \xi \), which may in turn be used as a source term in the equation:

\[
\xi = -\frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial^2 \psi}{\partial z^2} \frac{1}{r^2} \frac{\partial \psi}{\partial r} \quad \text{...(15)}
\]

(which arises from the definition of \( \xi \) and equation (8)) to determine a new field of \( \psi \) values. This last equation is solved using successive-over-relaxation techniques.

Whilst this method might be expected to be robust, it has been found so far that, in spite of the choice of a wide variety of densities and time steps, the method reduces major errors but does not eliminate the inequilibrium completely.
5 FINITE ELEMENT CORRECTION

The third scheme considered involved the use of finite element analysis. The stresses from the strain path method were passed to a finite element program and the equivalent out-of-balance forces calculated. These forces were then negated in a conventional incremental analysis. The corresponding displacements were considered as indicating the shift in the streamline pattern, and the updated streamlines were then used to compute a new strain path solution. Whilst not based on any rigorous interpretation, it was considered that this approach, which involves arriving at an equilibrium solution, albeit by an "incorrect" stress path, could lead to a workable correction scheme. The scheme did not in fact work, but this is thought to be at least in part due to difficulties in interpolation between the rectangular mesh used for the strain path method and the Gauss points in the triangular mesh used in the finite element analysis (this choice was for reasons beyond the scope of this paper). Even when no correction was carried out the two stages of interpolation resulted in significant stress changes.

It remains to be seen whether the method might be successful with more closely matched grids in the two numerical schemes.

6 STRAIN PATH FINITE ELEMENT METHOD

Instead of attempting to carry out iterative corrections to the strain path calculation, a different approach has been adopted, and has proved successful. The method involves combining the merits of the strain path method, which correctly accounts for steady flow but results in an error in equilibrium, with the finite element method which satisfies equilibrium correctly.

The problem of cone penetration has previously been analysed by finite element methods (e.g. de Borst and Vermeer (1984), Kiousis et al. (1987)). However, in these analyses the cone has effectively been introduced into a pre-bored hole, with the surrounding soil still in its in situ stress state. An incremental plastic collapse calculation is then carried out and the collapse value identified as the indentation pressure.

This interpretation is not entirely correct, however. During the real penetration of the cone, very high lateral and vertical stresses develop adjacent to the shaft of the cone. By influencing the stresses around the cone tip, these result in higher cone penetration pressures than are predicted by the analysis of a cone in a pre-bored hole. A careful distinction is necessary between a plastic collapse solution and a steady state penetration pressure.

In an attempt to solve the inequilibrium problem of the strain path method, whilst still accounting for the effects of continuous penetration, the following finite element analysis has been carried out. The solution from the strain path method is taken as the initial stress state. These stresses are not quite in equilibrium, with the inequilibrium being represented by a set of out of balance body forces. The cone is then held fixed while the out of balance forces are eliminated by incrementally applying equal and opposite forces. After the inequilibrium has been eliminated the cone is penetrated further until a steady load is reached. Because of the substantial displacements involved in this operation, a large strain formulation of the finite element method is used. Fifteen noded triangular elements are used, and the mesh is shown in Fig. 1.

Fig. 2 shows a comparison of three analyses using the finite element method, in which cone resistance is plotted against penetration. Curve AB is a conventional analysis starting from the in situ stresses, curve CD an analysis as described above, and EF a similar analysis but without correction of the inequilibrium. It can be seen that the equilibrium correction in fact has a very small effect on the ultimate resistance value. The lower resistance given by the analysis of a cone in a pre-bored hole is clearly apparent.

The in situ horizontal stress may be varied in the calculation. The variation of the cone factor with a measure of the horizontal stress $\Delta = (\sigma_{\text{vo}} - \sigma_{\text{ho}})/2\sigma_u$ where $-1 \leq \Delta \leq 1$ has been examined. The calculations indicate that $(q_{t} - \sigma_{\text{ho}})/\sigma_u$ is almost constant rather than $(q_{t} - \sigma_{\text{vo}})/\sigma_u$. Since, however, $\sigma_{\text{ho}}$ is not usually known accurately, it seems more practicable to retain the use of the factor $N_{kt} = (q_{t} - \sigma_{\text{vo}})/\sigma_u$, but to recognise the effect of the horizontal stress by using a cone factor which depends on $\Delta$.

Finite element analyses (for the pre-bored hole case) have been made of cones with smooth and rough shafts; the results are shown in Fig. 3. The cones on rough shafts show a slightly lower tip resistance since part of the vertical load is carried by friction on the shaft,
stress on the cone face due to surface roughness, defined as \( \alpha_f = \frac{\tau_f}{3/2s_u} \) where \( \tau_f \) is the shear stress on the cone face. There is a slight reduction due to the shaft roughness factor \( \alpha_s \) (defined similarly to \( \alpha_f \)).

![Fig. 1 Finite element mesh](image)

![Fig. 2 Load-penetration curves from finite element analysis](image)

Fig. 2 Load-penetration curves from finite element analysis although it is difficult to quantify the results from these analyses. Unfortunately, the roughness on the inclined cone face could not be varied in this study (in which all finite element analyses were for rough faced cones). More quantitative information on the effects of cone and shaft roughness has been obtained from a study of cones near the surface of a clay using the method of characteristics (Houlsby and Wroth, 1982). This study indicates an increase in mean stress.

The finite element analyses do not result in a precisely linear variation of \( N_{kt} \) with \( \ln(I_r) \), but it has been found that they can be very satisfactorily fitted by applying a simple factor to the solution for spherical cavity expansion. The final result is an approximate expression for \( N_{kt} \) which includes the effects of rigidity index, cone roughness and in situ stresses:

\[
N_{kt} = N_s \left( 1.25 + \frac{I_r}{2000} \right) + 2.4\alpha_f - 0.2\alpha_s - 1.8\Delta \tag{16}
\]

where \( N_s \) is the factor which appears in the spherical cavity expansion expression, i.e. \( N_s = \frac{4}{3} \left( 1 + \ln \left( \frac{I_r}{100} \right) \right) \). Eq.16 applies for the following ranges:

- \( I_r \) 50 to 500 (hence \( N_s \) 6.55 to 9.62)
- \( \Delta \) -1 to 1
- \( \alpha_f \) 0 to 1
- \( \alpha_s \) 0 to 1

In extreme combinations this results in \( N_{kt} \) factors from 6.4 to 18.6, but for more typical cases gives \( N_{kt} \) factors in the range 9 to 17. Although it is not
possible to test the full range of roughness cases, the above expression approximates the finite element calculations (13 runs) to better than 2% in every case.

Note that Eq. 16 requires estimates of the rigidity index and in situ horizontal stress (as well as the cone roughness) to be made before $q_t$ can be used to estimate $s_u$. The derived value of $s_u$ may then be used to refine the estimate of $I_r$. Whilst this procedure may be cumbersome, it is unavoidable if the influence of stiffness on $N_{kt}$ is to be accounted for.

Because the above analysis accounts for the high stresses which are developed on the cone shaft during penetration, it results in higher cone factors than have previously been derived theoretically. Since these higher cone factors are closer to those observed in practice, it is believed that the analysis is reasonably realistic.

Note that the penetration analysis demonstrates the important influence of the rigidity index $I_r$ on the interpretation of the cone. Whilst this may be inconvenient, it cannot be ignored. If the CPT is to be rationally interpreted then information about soil stiffness is required.

7 CONCLUSIONS

The penetration of a cone into clay under undrained conditions has been analysed using the strain path method. The resulting stress field is not quite in equilibrium, and various methods have been explored to achieve a correction to the flow field which would eliminate the equilibrium. None of the methods attempted was entirely successful. An alternative approach in which the results of the strain path analysis are used as the starting point for a finite element analysis has been successful.

For a cone penetrating an elastic-perfectly plastic material theoretical values of cone factors have been derived. The cone factor depends on:

a. The rigidity index
b. The horizontal stress
c. Roughness of cone and shaft.

An approximate expression for the cone factor in terms of these parameters has been proposed (Eq. 16).

ACKNOWLEDGEMENTS

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REFERENCES

Analysis of the Piezocone in Clay

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ABSTRACT: An analysis is presented of the penetration of a CPT into clay, followed by the dissipation of pore pressures around the cone. The analysis is based on the Strain Path Method, with additional detail provided by large strain finite element analysis. The primary importance of the rigidity index and of the horizontal stress in affecting the cone factor $N_{kt}$ is discussed. Based on finite difference analysis of the consolidation phase, a new interpretation method for consolidation data from piezocones is suggested.

Keywords: analysis, clays, cone factor, consolidation, finite elements, piezocone, strain path

1 INTRODUCTION

The purpose of this paper is to present the results of an analysis of the piezocone in clay. Whilst the main features of the analysis are presented here, the intention is to concentrate on the numerical results and their implications for the interpretation of piezocone data. Full details of the analysis are given by Teh (1987) and a summary is given by Teh and Houltsby (1988).

The analysis consists of two phases. In the first the penetration of a CPT into clay is analysed, with the clay idealised as an incompressible elastic-perfectly plastic (von Mises) material. Values of the cone factor $N_{kt}$ (defined as $(q_t - \sigma_0)/s_u$) are derived. In the second stage the consolidation around a static cone is analysed using uncoupled Terzaghi-Rendulic consolidation theory, and methods of interpretation of the data from dissipation tests are suggested.

2 DERIVATION OF $N_{kt}$ FACTORS

Approximate derivations of $N_{kt}$ factors have in the past been based mainly on either bearing capacity theory or cavity expansion theory. Modification of the results of plane strain bearing capacity calculations for application to axially symmetric problems is often carried out empirically, although numerical solutions to axisymmetric problems can be obtained using plasticity theory (e.g., Cox et al. (1961), Houltsby and Wroth (1983)). Adaptation of bearing capacity theory to deep penetration problems is much more difficult, and either involves adoption of boundary conditions which are not appropriate for real cone penetrometers (e.g., Kouralt and Kaku (1982)) or results in $N_{kt}$ factors which increase indefinitely with increased penetration (Houltsby and Wroth, 1982). In practice empirical depth factors are applied to shallow bearing capacity solutions.

The problem of the transition between shallow and deep penetration is readily explained by the fact that shallow penetration involves a mechanism in which displaced material moves entirely outwards to the free surface, whilst in deep penetration the displaced material is accommodated by elastic deformation of the soil. Thus cavity expansion theories are applied to deep penetration problems (Vesic, 1972). It is widely held that the limit solution for the cylindrical cavity expansion pressure $\psi_c$ is applicable to estimation of the radial stress on the shaft of a penetrometer. For the von Mises yield criterion:
the solution for cylindrical cavity expansion is:

\[ \psi_c = \sigma_{ro} + \frac{2s_u^2}{\sqrt{3}} \left( 1 + \ln \frac{\sqrt{3} I_r}{2} \right) \]  \hspace{1cm} (2)

where \( I_r \) is the rigidity index \( G/s_u \).

Note that \( s_u \) is the undrained strength in triaxial compression. The von Mises criterion implies a shear strength in plane strain of \( 2s_u/\sqrt{3} \).

The spherical cavity expansion pressure \( \psi_s \) is often considered as an estimate of the pressure at the cone tip:

\[ \psi_s = \sigma_{ro} + \frac{4s_u^2}{3} \left( 1 + \ln I_r \right) \]  \hspace{1cm} (3)

Note that both of these expressions demonstrate the important influence of the rigidity index through the terms in \( \ln(I_r) \).

Baligh (1986) has pointed out that an inconsistency in the application of cavity expansion theory to cone penetration problems is that it does not model correctly the strain paths followed by soil elements. He suggests the use of the Strain Path Method for problems of deep penetration, and this method has been used in this research.

By changing the frame of reference, the penetration of a cone in a homogeneous material can be viewed as a steady state flow of soil past a static penetrometer. An initial estimate of the streamlines for the flow can be made, for instance by using inviscid flow theory. The flow pattern will of course be slightly in error since an incorrect constitutive law has been used, but because the problem is heavily constrained kinematically the estimate represents a reasonable first approximation. From this flow pattern the complete history of strain for each soil element may be determined. Using appropriate initial stresses, together with the elastic-plastic constitutive laws (as described by Houslsby et al. (1985)), it is possible to calculate numerically the deviatoric stresses in the soil by integrating along the streamlines.

The mean normal stress may then be found by using one of the equilibrium relations (radial or axial). It is then found that the stresses do not exactly obey the other equilibrium relationship, with the discrepancy reflecting the error in the initial flow field. In principle the other equilibrium relationship could be used to correct the flow field, but in practice no fully satisfactory means of achieving this has yet been found (in spite of attempts using various plausible numerical schemes).

Nevertheless, if the radial equilibrium equation is used to derive the mean stress, the in equilibrium can be shown to be relatively small and confined mainly to a very small region near the cone tip and a slightly larger region near the cone shoulder. The results of such calculations for cone factors for 60° cones in a soil with an initially isotropic stress state are shown in Fig. 1, where it is clear that the variation of \( N_{kt} \) with \( I_r \) shows the same trend as is predicted by cavity expansion theory. The values can be approximated by the expression:

\[ N_{kt} = 1.25 + 1.84 \ln(I_r) \]  \hspace{1cm} (4)

In carrying out the above calculation no control is possible of the shear stress boundary condition on either the cone face or the shaft. The results are found to correspond to a cone roughness factor \( \alpha_f = \sqrt{3} \tau_f/2s_u \) (where \( 0 \leq \alpha_f \leq 1 \)) of approximately zero, where \( \tau_f \) is the shear stress on the cone face. If roughness variation is assumed to have no significant effect on the normal stress distribution on the face of the cone, then it can be accounted for entirely by considering the additional vertical force component due to the shear stress. This
results in an additional term on the cone factor equal to $2\alpha_f/\sqrt{3}\tan(\beta/2)$ where $\beta$ is the cone angle. For the conventional 60° cone this conveniently reduces to $2\alpha_f$, so that Eq. 4 becomes modified to:

$$N_{kt} = 1.25 + 1.84 \ln(I_t) + 2\alpha_f \quad \ldots \ldots (5)$$

The in situ horizontal stress may be varied in the calculation. Fig. 2 shows the variation of the cone factor with a measure of the horizontal stress $\Delta = (\sigma_{vo} - \sigma_{ho})/2s_u$ (where $-1 \leq \Delta \leq 1$). The calculations fall very close to a line of slope -2 on this plot, which corresponds to the result that $(q_t - \sigma_{ho})/s_u$ is almost constant rather than $(\sigma_t - \sigma_{vo})/s_u$. Since, however, $\sigma_{ho}$ is not usually known accurately, it seems more practicable to retain the use of the factor $N_{kt} = (q_t - \sigma_{vo})/s_u$, but to recognise the effect of the horizontal stress by further modifying Eq. 4 to:

$$N_{kt} = 1.25 + 1.84 \ln(I_t) + 2\alpha_f - 2\Delta \quad \ldots \ldots (6)$$

![Fig. 2 Variation of cone factor with $\Delta$](image)

Whilst the above approximate solutions are of value, there remains a doubt about the significance of the inequilibrium of the stresses. In order to resolve this problem an alternative series of calculations has been carried out using incremental large displacement finite element analysis.

2 FINITE ELEMENT METHOD

The problem of cone penetration has previously been analysed by finite element methods (e.g. de Borst and Vermeer (1984), Kiousis et al. (1987)). However, in these analyses the cone has effectively been introduced into a pre-bored hole, with the surrounding soil still in its in situ stress state. An incremental plastic collapse calculation is then carried out and the collapse value identified as the indentation pressure.

This interpretation is not entirely correct, however. During the real penetration of the cone, very high lateral and vertical stresses develop adjacent to the shaft of the cone. By influencing the stresses around the cone tip, these result in higher cone penetration pressures than are predicted by the analysis of a cone in a pre-bored hole. A careful distinction is necessary between a plastic collapse solution and a steady state penetration pressure.

In an attempt to solve the inequilibrium problem of the strain path method, whilst still accounting for the effects of continuous penetration, the following finite element analysis has been carried out. The solution from the strain path method is taken as the initial stress state. These stresses are not quite in equilibrium, with the inequilibrium being represented by a set of out of balance body forces. The cone is then held fixed while the out of balance forces are eliminated by incrementally applying equal and opposite forces. After the inequilibrium has been eliminated the cone is penetrated further until a steady load is reached. Because of the substantial displacements involved in this operation, a large strain formulation of the finite element method is used. Fifteen noded triangular elements are used, and the mesh is shown in Fig. 3.

![Fig. 3](image)

Fig. 4 shows a comparison of three analyses using the finite element method, in which cone resistance is plotted against penetration. Curve AB is a conventional analysis starting from the in situ stresses, curve CD an analysis as described above, and EF a similar analysis but without correction of the inequilibrium. It can be seen that the equilibrium correction in fact has a very small effect on the ultimate resistance value. The lower resistance given by the analysis of a cone in a pre-bored hole is clearly apparent.

For different combinations of initial horizontal and vertical stresses the same trend of variation of $N_{kt}$ with $\Delta$ as shown in Fig. 2 is observed, but with a slope of approximately -1.8 rather than -2.0.

Finite element analyses (for the pre-bored hole case) have been made of cones
inclined cone face could not be varied in this study (in which all finite element analyses were for rough faced cones). More quantitative information on the effects of cone and shaft roughness has been obtained from a study of cones near the surface of a clay using the method of characteristics (Houlsby and Wroth, 1982). This study indicates a slight increase in mean stress on the cone face due to surface roughness, suggesting a factor of $2.4\alpha_f$ rather than $2\alpha_f$ as in Eq. 5. The reduction due to the shaft roughness factor $\alpha_s$ (defined similarly to $\alpha_f$) can be approximated by a further term of $-0.2\alpha_s$.

Fig. 3 Finite element mesh

Fig. 4 Load-penetration curves from finite element analysis

with smooth and rough shafts; the results are shown in Fig. 5. The cones on rough shafts show a slightly lower tip resistance since part of the vertical load is carried by friction on the shaft, although it is difficult to quantify the results from these analyses. Unfortunately the roughness on the

Fig. 5 Finite element analyses of rough cones with smooth and rough shafts

The finite element analyses do not result in a precisely linear variation of $N_{kt}$ with $\ln(I_r)$, but it has been found that they can be very satisfactorily fitted by applying a simple factor to the solution for spherical cavity expansion. The final result is an approximate expression for $N_{kt}$ which includes the effects of rigidity index, cone roughness and in situ stresses:

$$N_{kt} = N_s (1.25 + \frac{I_r}{2000}) + 2.4\alpha_f - 0.2\alpha_s - 1.8\Delta$$  \hspace{1cm} ...(7)

where $N_s$ is the factor which appears in the spherical cavity expansion expression, i.e. $N_s = \frac{4}{3}(1 + \ln[I_r])$. Eq. 7 applies for the following ranges:
I_r 50 to 500 (hence \( N_s \) 6.55 to 9.62)
\[ \Delta \] -1 to 1
\[ \alpha \] 0 to 1
\[ \alpha_s \] 0 to 1

In extreme combinations this results in \( N_{kt} \) factors from 6.4 to 18.6, but for more typical cases gives \( N_{kt} \) factors in the range 9 to 17. Although it is not possible to test the full range of roughness cases, the above expression approximates the finite element calculations (13 runs) to better than 2% in every case.

Note that Eq. 7 requires estimates of the rigidity index and in situ horizontal stress (as well as the cone roughness) to be made before \( q_t \) can be used to estimate \( s_u \). The derived value of \( s_u \) may then be used to refine the estimate of \( I_r \). Whilst this procedure may be cumbersome, it is unavoidable if the influence of stiffness on \( N_{kt} \) is to be accounted for.

Because the above analysis accounts for the high stresses which are developed on the cone shaft during penetration, it results in higher cone factors than have previously been derived theoretically. Since these higher cone factors are closer to those observed in practice, it is believed that the analysis is reasonably realistic.

3 CONSOLIDATION ANALYSIS

The results of the above analysis, together with an empirical formula for estimating changes in pore pressure (Henkel, 1959):

\[ \Delta u = \Delta \sigma_{oct} + \alpha \Delta \tau_{oct} \quad \ldots \quad (8) \]

have been used to estimate pore pressures during penetration of the cone. These are then used as the starting point for a consolidation analysis using uncoupled Terzaghi-Reidulic consolidation theory, solved using an Alternating Direction Implicit finite difference scheme.

The effect of \( \alpha \) in Eq. 8 is relatively small, and excess pressures are primarily due to changes in mean stress. The initial pore pressure distribution for \( I_r = 100 \), and the pore pressures after dissipation to a time factor \( T = c_h t/R^2 = 1.0 \) are shown in Fig. 6.

Note that the intense gradient of initial pore pressure near the cone shoulder may indicate a disadvantage in placing the piezometric element at this location. Calculations using different ratios of \( c_h/c_v \) show that for \( c_h > c_v \) consolidation behaviour is dominated by the value of \( c_h \), with \( c_v \) having little effect.

Fig. 7 shows the non-dimensional dissipation curves for excess pore pressure against time (on a logarithmic scale) for six different points along the penetrometer (numbered 1 to 6 as cone tip, half way up the face, cone shoulder, 5, 10 and 20 radii above the shoulder). The shapes of the curves are exactly as would be expected, except for the dissipation at the cone tip, which shows a more unusual response due to the redistribution of pore pressure from further up the cone.

These dissipation curves are not, unfortunately, unique, since the initial pore pressure distribution depends on the value of \( I_r \). In particular the excess
pore pressures develop mainly in the plastically deforming region, which is of an approximate radius proportional to \(\sqrt{I_r}\).

It may therefore be more rational (although less convenient) to choose a dimension related to the size of the plastically deforming zone, i.e. \(R/I_r\) rather than the radius of the cone \(R\) as the quantity used in deriving the dimensionless time factor. This would result in a definition of time factor of \(c_h t/R^2/\sqrt{I_r}\). In fact the most satisfactory way of unifying the results at different \(I_r\) values has been found to be a compromise between the two extreme possibilities. If the time factor is defined as \(T^* = c_h t/R^2/\sqrt{I_r}\) then the dissipation curves for all values of \(I_r\) from 50 to 500 are very satisfactorily unified, as shown for instance for pore pressure dissipation at a point half way up the cone face in Fig. 8. Using these results the modified time factors shown in Table 1 are obtained.

**Fig. 8** Excess pore water pressure dissipation against modified time factor \(T^*\)

**Table 1, Modified time factors \(T^*\) from consolidation analysis**

<table>
<thead>
<tr>
<th>Degree of consol.</th>
<th>Tip face</th>
<th>Cone face</th>
<th>Shoulder 5 radii</th>
<th>Above above shoul.</th>
<th>5 rad.</th>
<th>10 rad. above</th>
<th>Above above shoul.</th>
</tr>
</thead>
<tbody>
<tr>
<td>20%</td>
<td>0.001</td>
<td>0.014</td>
<td>0.038</td>
<td>0.294</td>
<td>0.378</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30%</td>
<td>0.006</td>
<td>0.032</td>
<td>0.078</td>
<td>0.503</td>
<td>0.662</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40%</td>
<td>0.027</td>
<td>0.063</td>
<td>0.142</td>
<td>0.756</td>
<td>0.995</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50%</td>
<td>0.069</td>
<td>0.118</td>
<td>0.245</td>
<td>1.11</td>
<td>1.46</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60%</td>
<td>0.154</td>
<td>0.226</td>
<td>0.439</td>
<td>1.65</td>
<td>2.14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>70%</td>
<td>0.345</td>
<td>0.463</td>
<td>0.804</td>
<td>2.43</td>
<td>3.24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>80%</td>
<td>0.829</td>
<td>1.04</td>
<td>1.60</td>
<td>4.10</td>
<td>5.24</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The figures in Table 1 have been compared with previous analytical studies of consolidation which were based on cylindrical cavity expansion (uncoupled analysis by Torstensson (1977) and coupled analysis by Randolph and Wroth (1979)).

Taking due account of the different values of \(G/s\) used in the analyses, it is found that the consolidation rate calculated at the shoulder is very similar to the cylindrical analysis. At the tip the new analysis predicts faster dissipation, and on the shaft (5 and 10 radii behind the shoulder) the dissipation is predicted to be considerably slower than that given by the cylindrical theory. The explanation lies in the fact that the gradient of mean stress with radius near the cone shoulder compares closely with cylindrical cavity expansion theory (Teh, 1987). On the shaft the initial pore pressures are lower, and the gradient with radius much flatter. The consolidation is further delayed by redistribution of high pore pressures from the tip region.

Another useful way of interpreting consolidation data is the root time plot, where the initial section of the plot usually approximates closely to a straight line. This is indeed found to be the case, and the slope of the plot of degree of consolidation against \(\sqrt{T^*}\) is given in Table 2, which is applicable for \(I_r\) in the range 50 to 500.

**Table 2, Initial slopes for dimensionless root time plots**

<table>
<thead>
<tr>
<th>Location</th>
<th>Tip face</th>
<th>Cone face</th>
<th>Shoulder 5 radii</th>
<th>Above above shoul.</th>
<th>5 rad.</th>
<th>10 rad. above</th>
<th>Above above shoul.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope</td>
<td>1.30</td>
<td>1.63</td>
<td>1.15</td>
<td>0.62</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note that both the penetration analysis and the consolidation analysis demonstrate the important influence of the rigidity index \(I_r\) on the interpretation of the cone. Whilst this may be inconvenient, it cannot be ignored. If the CPT is to be rationally interpreted then information about soil stiffness is required.

4 CONCLUSIONS

For a cone penetrating an elastic-perfectly plastic material theoretical values of cone factors have been derived. The cone factor depends on:

a. The rigidity index
b. The horizontal stress
c. Roughness of cone and shaft.

An approximate expression for the cone factor in terms of these parameters has been proposed (Eq. 6).
Dissipation of pore pressure around the cone is primarily governed by the horizontal permeability. Interpretation of pore pressure dissipation curves depends on the location of the piezometric element. An almost unique interpretation of the curves can only be achieved if the dimensionless time factor accounts for the effect of rigidity index. Time factors at different degrees of consolidation are suggested.

ACKNOWLEDGEMENTS

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Undrained deep penetration, I: shear stresses

M. M. BALIGH (1986). Géotechnique 36, No. 4, 471-485

C. T. Teh and G. T. Houlsby, Oxford University
The Author has presented an interesting analysis of the undrained deep penetration problem using the strain path method. The simple pile solution has clearly demonstrated the two-dimensional nature of the deep penetration problem which is ignored by the cavity expansion approach. The result revealed some important characteristics of this problem including strain and stress reversals which are not evident in the cavity expansion solution.

The Author derived the stress path in the elastic zone from the closed form solution (Baligh, 1985) for strains. These analytical expressions for strains were based on the simplifying assumption that the streamlines are parallel to the pile axis. Thus, the solution is approximate and applies only in the region where the streamlines are nearly straight. However, for a soil with a low stiffness-to-strength ratio, the elastic-plastic boundary may lie in a region close to the pile where the streamlines are significantly curved. Clearly then, derivation of the stress path based on a simplified strain expression would not be appropriate. By taking into consideration the profile of the streamlines, the Writers have obtained the exact analytical expressions for strains around a simple pile.

The geometry of the problem is illustrated in Fig. 1.

**STRAIN PATH**
The strain $\varepsilon_{ij}$ at any point along the streamline is defined as

$$\varepsilon_{ij} = \int \epsilon_{ij} \, dt + \varepsilon_0 \quad (1)$$

where $\epsilon_{ij}$ is the strain rate tensor. The in situ strain in the soil, $\varepsilon_0$, can be assumed to be zero without loss of generality. Then

$$\varepsilon_{ij} = \int \dot{\varepsilon}_{ij} \, dt = \int \dot{\varepsilon}_{ij} \frac{dS}{V} \quad (2)$$

where $dS$ is the infinitesimal arc length along the streamline and $V$ is the velocity of the soil element. For the simple pile, equation (2) can be transformed into a function of $\phi$ only. Integrating equation (2) along the streamline yields the following analytical expressions

$$\varepsilon_1 = F_1(\phi) + (1 - \frac{1}{2}B^2) F_2(\phi) \quad (3)$$
$$\varepsilon_2 = -F_1(\phi) - \frac{1}{2}(1 - 3B^2) F_2(\phi) \quad (4)$$
$$\varepsilon_\theta = -\frac{1}{2} F_2(\phi) \quad (5)$$
$$\varepsilon_{rs} = -\frac{3}{2} \left[ 3B^2 - 1 \right] \left( \frac{\pi - \phi}{2} \right) + B \sin \phi \right]$$

$$- \frac{\sin (2\phi)}{4} - 2B(B^2 - 1)^{0.5} \times \tan^{-1} \left[ \left( \frac{B + 1}{B - 1} \right)^{0.5} \cot \left( \frac{\phi}{2} \right) \right] \quad (6)$$

where

$$F_1(\phi) = 3(1 + \cos \phi) \left[ \left( \frac{r_0}{R} \right)^2 + \frac{3}{4} - \frac{\cos \phi}{4} \right] \quad (7)$$
$$F_2(\phi) = \ln \left[ 1 + \frac{1}{2} \left( \frac{r_0}{R} \right)^2 (1 + \cos \phi) \right] \quad (8)$$
$$B = 2 \left( \frac{r_0}{R} \right)^2 + 1 \quad (9)$$

![Fig. 1. Simple pile problem: geometry of the strain path solution](image-url)
Fig. 2. Strain paths in the near field of a simple pile

*R* is the limit radius of the simple pile and \( r_0 \) is the initial radial distance of the streamline. Equations (3)–(6) are applicable in both the far field and the near field of the pile. When the initial location of the streamline is very large (\( r_0 \gg 1 \)) but not infinite, the leading order terms of these strain components can be shown to be identical with the approximate far field solutions given by the Author.

When equations (3)–(6) are plotted using the strain deviators \( E_1, E_2, \) and \( E_3 \) as defined by the Author, the strain paths in the far field are virtually identical with those given by the Author. However, as \( r_0/R \) decreases (close to the pile shaft), the projections of the strain path on the \( E_i \) plane (Fig. 2) take on a progressively skewed appearance. The results obtained numerically by Baligh (1985) show the same trend. The following points are evident from Fig. 2:

(a) in both the near field and the far field, \( E_2 \) increases monotonically, but \( E_1 \) and \( E_3 \) exhibit significant strain reversals.

(b) in the near field, as \( z \to \infty (\phi \to 0) \), both \( E_1 \) and \( E_3 \) are no longer zero (as in the far field) but have finite residual values.

**STRESS PATH**

Given the strain path, the stress path (including the initial elastic response) can be derived by adopting a suitable constitutive relationship for the soil. For comparison, the Writers adopted the same bilinear Prandtl–Reuss model with the Von Mises yield criterion as used by the Author. The strain increment between any two points on the streamline can be determined exactly using equation (3). The stress–strain relationship has been accurately integrated using small strain increments and a closed form solution due to Booker (1984). This solution relates the stress increment due to a corresponding strain increment and is applicable only to the zero rotation rate case, which applies for the assumed inviscid flow strain rate solution.

For a soil with \( G/k = 100 \), the stress paths thus obtained are plotted in Figs 3 and 4. Fig. 3 shows the stress paths for soil elements initially located at radial distances \( R, 5R, 10R \) and \( 15R \) from the pile axis. These results are comparable with those given by the Author. In the inner plastic zone, however, there is a significant difference between the Writers’ results and the Author’s limit solution. Fig. 4 shows the stress path of a soil element...
initially located at $r_0 = 0.01 R$. Far behind the pile tip (as $z \to \infty$, $\phi \to 0$), the stress deviator $S_3$ has a finite value. This should be compared with the Author's solution which indicates that $S_3 = 0$ on the shaft.

The present solution shows that the shear stress in the immediate vicinity of the pile shaft is not zero even though the strain field is based on an inviscid flow solution. This stress state therefore does not correspond to a simple pile with a smooth shaft as the Author's result would suggest. One possible explanation for the difference in $S_3$ might be the inclusion of the initial elastic response in the stress calculation. The Writers recognize that the strain path method yields only an approximate solution because the equilibrium equations are not fully satisfied. Nevertheless, it is essential to resolve the discrepancy regarding the value of $S_3$ close to the shaft before the result can be used to interpret pile installation and deep penetration problems in a meaningful way.

The Writers have also analysed the problem of a cone penetrometer with a $60^\circ$ tip using a numerical approach. The stress paths in the far field of the penetrometer are virtually identical with those of the simple pile. This indicates that the simple pile solution can be used to estimate the stress state in the far field of a cone penetrometer. However, there are significant differences in the near field solutions. Fig. 5 shows the stress path of a soil element initially located at $r_0/R = 1.0$ from the axis of the penetrometer. A comparison of Figs 3 and 5 shows that near field stress paths are significantly affected by the shape of the penetrometer. It is therefore reasonable to conclude that in a rational analysis of the penetrometer the actual geometry of the cone (more specifically, the cone angle) must be taken into consideration.

REFERENCES


Elastic-plastic incompressible flow around an infinite cone

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ABSTRACT: A solution is presented for the incompressible soil flow around an infinite cone. It can be of application for cone penetration analyses in clay, for the region around the cone tip, which is a singular point. The soil is considered as incompressible, elastoplastic, with Von Mises yield condition. The influence of cone angle and roughness as well as soil rigidity index, are analyzed.

1 INTRODUCTION

In the analysis of the stress and velocity fields during quasi-static cone penetration in clay, the region near the cone tip plays an important role. If a numerical technique is used for the solution, the tip is a singular point (Fig. 1) in which some of the variables become infinite and some others have a discontinuity, leading to numerical problems. The aim of this paper is to obtain a solution for this region which can be used as a boundary condition for numerical analyses, avoiding the actual singularity.

On the other hand, many of the features of the penetration of an actual finite cone are present near the tip. The simplification of the problem geometry from Fig. 1-a) to Fig. 1-b) allows for close-form solutions and hence the identification of these features and of the factors controlling them. For instance, the two possibilities for the plastic zone of Fig. 1-c) lead, in the tip region, to all the soil in plastic state or to an elastic zone ahead the tip limited by a radial elastic-plastic boundary.

In this paper, an analysis of the tip region is presented, by considering it as an infinite cone penetrating at a constant velocity, $v_0$, into a clay space. In order to have a steady state problem, the process is studied as viewed from the cone, i.e., taking the cone as fixed and the soil flowing upwards around it (Fig. 1-b). The theoretical formulation is only briefly described, whilst the results are presented in more detail.

There are some published analyses of this problem, referring to metal indentation:

![Fig. 1. Penetration problem. Tip region](image)
The soil is treated as incompressible and elastoplastic with the Von Mises yield condition, defined by a shear modulus, $G$, and a shear strength, $s_s$ in triaxial conditions ($\sigma_2=s_3$). The ratio $I_r=G/s_s$ is called "rigidity index".

The problem unknowns are the soil velocities and stresses, shown in Fig. 2-a). Due to the axial symmetry, spherical coordinates are used, and the hoop angle ($\phi$) is not appearing in the formulation. The stresses (\(\sigma_{ij}\)) are separated into their isotropic ($p$) and deviatoric (\(\sigma'_{ij}\)) components. By using the constitutive relationships described by Houlsby et al. (1985), the governing equations are, neglecting the Jaumann terms in the stress rate relationships:

\[ \varepsilon_{vol} = -\frac{\partial v}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial \theta} - 2\frac{v}{r} - \frac{u}{r}\cot\theta = 0 \quad (1) \]

- Stress-strain rate relationships:

\[ \varepsilon_r = \frac{1}{2}\frac{\partial \sigma_r}{\partial r} + 6\lambda\sigma_r = -\frac{\partial v}{\partial r} \quad (2) \]

\[ \varepsilon_{\theta} = \frac{1}{2}\frac{\partial \sigma_{\theta}}{\partial \theta} + 6\lambda\sigma_r = -\frac{\partial u}{\partial \theta} - \frac{v}{r} \quad (3) \]

\[ \gamma_{\theta r} = \frac{1}{2}\frac{\partial \sigma_{\theta r}}{\partial \theta} + 12\lambda\sigma_r = \frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{u}{r} \quad (4) \]

- Equilibrium:

\[ \frac{\partial p}{\partial r} - \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{1}{r}\tau_{\theta r}\cot\theta + \frac{3}{r}\sigma_r = 0 \quad (5) \]

\[ \frac{\partial p}{\partial \theta} + \frac{\partial \tau_{\theta r}}{\partial r} - 3\tau_{\theta r}\cot\theta + (\sigma'_{rr} + 2\sigma'_{\theta\theta})\cot\theta = 0 \quad (6) \]

- Elastic/plastic state:

\[
\lambda = 0 \quad \text{ if } F < 0 \\
\lambda > 0 \quad \text{ if } F = 0 
\]

where:

\[ F = \sigma'_{r}^{2} + \sigma'_{\theta} + \sigma'_{\theta}^{2} + \tau_{r\theta}^{2} - \frac{4}{3}s_{s}^{2} \]

A detailed analysis of the above equations leads to the following conclusions:

i) The deviatoric stresses ('\(\sigma'_{ij}\)) can be considered constant with the radial coordinate, $r$.

A further analysis of the equations shows that if there is an elastic zone the velocities cannot vary with the distance to the tip ($n=0$ in eq. 10). Hence, can be concluded that there are two possible situations:

i) All the soil is in plastic state (elastic zone), and the power '$n$' is zero. This will be called PLASTIC CASE.

ii) There exists an elastic zone before the tip and a plastic zone around the nose. The power '$n$' is zero. This will be called ELASTOPLASTIC CASE.

So, the unknown variables, denoted with an upper bar ('^') are functions of the angle $\theta$ only, and the problem is governed by a system of ordinary differential equations. The parameters $\nu'$ (eq. 9) and $c_1$ (eq. 10) and the integration constant must be obtained from the pertinent boundary conditions, which are as follows:

i) At the axis ($\theta = 0$), the axial symmetry conditions ($u = 0$, $\tau_{r\theta} = 0$, $\sigma'_{r} = -\sigma'_{\theta} = -\sigma'_{rr} = 0$) imposed velocity ($v = v_{0}$) and pressure $p$.
ii) At the cone face ($\theta=\theta_1$), the conditions of no normal velocity ($v=0$) and roughness ($f_{g2}=2\sqrt{2}a\cdot s_0$), where $\alpha (0<\alpha<1)$ is the adhesion factor of the cone face.

The physical meaning of the foregoing derivation is that if the values of the upwards velocity ($v_0$) and the pressure ($p_0$) are imposed at a point $A$ (Fig. 2-b) located at some distance $r_0$ along the axis, then the integration of the resulting system of ordinary differential equations with respect to $\theta$ provides the variation of the stresses and velocities around the tip at any point $M$ along the arc $A$-$B$. Their values at any inner point $M'$ can be obtained from the values at $M$ using the equations (9) and (10), once the radial variation parameters for the velocities ($n$) and for the pressure ($\nu$) are known.

3 ANALYSIS OF RESULTS

3.1 General

The above equations have been solved for a reasonably complete range of soil properties ($I_p$) and cone characteristics ($B, \alpha$).

Fig. 3 shows, as an example, the flow pattern around the tip of a standard cone ($B=30^\circ$), of a soil with a rigidity index, $I_p=100$, in the two extreme cases of smooth ($\alpha=0$) and rough ($\alpha=1$) cone face. The solution is of the PLASTIC CASE type (i.e., all the soil is in plastic state). The power "n" (eqs. 10) is also shown. Fig. 4 shows the principal deviatoric stress field.

In Fig. 5, the contours for the pressure increment, $(p-p_0)/s_0$, have been plotted. The value of the pressure radial variation parameter, $\nu$, is also shown.

3.2 Limit between elastoplastic and plastic cases

The first general meaningful result refers to the range of the ELASTOPLASTIC and PLASTIC cases.
Fig. 6 Limit cone angle between elastoplastic and plastic cases

TIC CASES. As a general fact, the existence of an elastic zone (ELASTOPLASTIC CASE) is only possible for very sharp cones (small values of $\theta$). Fig. 6 shows the critical value of $\theta$ for the transition between the two cases plotted against the soil rigidity index, for different cone roughness factors. As it can be seen, for real soils ($I_r$ of 50-500), the standard cones fall well into the PLASTIC CASE range, the limit value being about $5^\circ-10^\circ$. A notable exception are the early piezoprobe (Torstensson, 1975; Wizza et al., 1975), which had values of $\theta$ of $5^\circ$ and $10^\circ$, respectively. It can be proved theoretically that for rigid-plastic material ($I_r=\infty$), the ELASTOPLASTIC CASE is not possible ($\beta=0$), in agreement with the trend of the curves of Fig. 6.

It is worth noting that some authors have found, as a result from numerical analyses, the difference between the two described cases. For instance, Baligh (1985) states that a blunt ($\theta=30^\circ$) cone involves "large deformations that are clearly visible in the vicinity of the tip", while a sharp ($\theta=5^\circ$) cone acts in a different way, "cutting instead of pushing the soil ahead of the cone". These qualitative comments are expressed by the power "n" being positive or zero, respectively.

From these considerations, the results for the ELASTOPLASTIC CASE are not detailed further, restricting the presentation to the PLASTIC CASE.

### Table 1. Results for the case $\theta=30^\circ$ (standard cone).

<table>
<thead>
<tr>
<th>$I_r$</th>
<th>$\alpha$</th>
<th>$n$</th>
<th>$\nu$</th>
<th>$\bar{V}_c$</th>
<th>$\bar{p}$</th>
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<tr>
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<tr>
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<td>-0.634</td>
<td>0.825</td>
<td>3.5</td>
</tr>
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<tr>
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<tr>
<td>200</td>
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<td>0.056</td>
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<tr>
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<td>3.0</td>
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<tr>
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### 3.3 Results for the plastic case

The main magnitudes defining the solution are the parameters governing the radiation of the pressure ($\nu$) and of the velocities ($n$). It is also of interest to know the stresses and velocities at the cone face (denoted from here on with the script "c"). The velocities are given by the exit velocity ratio $\bar{V}_c$. The pressure increment is defined by $\bar{p}_c$.

For the deviatoric stresses in the plastic zone, Csuk's variables $\xi$ and $\psi$ are used, defined by:

\[
\sigma'_{rr} = \frac{4}{3} \cos \xi \cdot \cos \psi
\]

\[
\sigma'_{\theta} = -\frac{4}{3} \cos \xi \cdot \sin(\psi + \pi/6)
\]

\[
\bar{\tau}_{r\theta} = \frac{2}{\sqrt{3}} \sin \xi
\]

Their values at the cone face can be found from the boundary conditions.
\[
\zeta_c = -\sin^{-1} \alpha \\
\psi_c = -\tan^{-1} \left( \frac{n \cdot 2}{n^3} \right)
\]  
(12)

Table 1 shows the obtained values of \( n' \), \( \psi' \), \( \psi \), and \( \rho \), for a standard cone \((\beta = 30^\circ)\), for values of the rigidity index covering the range 10-1000, and for three different values of the cone roughness, \( \Delta \). As can be seen, a soil with \( I > 200 \) can be considered as rigid-plastic for practical purposes. The power \( n' \) is small, about 0.05 for a lubricated cone and 0.16 for a rough cone. The exit velocity ratio is less than one, decreasing with the cone roughness, from about 0.82 to about 0.28. The pressure parameter \( \psi' \) is negative, about -0.60, implying that at the tip there are infinite compressions (see eq. 9), the influence of the roughness being small. The pressure increment is positive, decreasing with \( \Delta \), from 3.80 to 2.80.

The influence of the cone angle, \( \beta \), has been investigated for a typical case of a soil with a rigidity index of 100, which can be considered as an average value for real soils. Figs. 7 to 10 show the results for different cone roughness factors.

The extreme case of a lubricated flat-punch \((\beta = 90^\circ, \Delta = 0)\) has a simple analytical solution, for any rigidity index, because it corresponds to a homogeneous incompressible deformation under conventional triaxial compression \((\sigma_1 = \sigma_2 = \sigma_3)\), the solution being given by \( n = 1, \psi = 0, \psi' = 0.5 \) and \( \rho = 0 \). This case has been included in the figure and provides a check of the numerical solution obtained for other cases.

The power \( n' \) increases with the cone angle \( \beta \) and roughness \( \Delta \) (Fig. 7). For lubricated cone \((\alpha = 0)\), it reaches the value of \( n = 1.0 \) for \( \beta = 90^\circ \), whilst for a rough cone \((\alpha = 1)\) this happens at an angle of about 50°.

The pressure parameter \( \psi' \) is shown in Fig. 8. The influence of the cone roughness is small, and of varying trend. It is noted that for blunt roughness \((\beta > 60^\circ)\) it becomes positive, what implies the presence of infinite suction (see eq. 9) instead of compressions at tip.

The exit velocity ratio (Fig. 9) decreases with the cone angle and roughness. For blunt cone, it becomes zero at an angle of about 60°. This implies that for blunt cones, the action of the full face would produce a negative exit velocity (i.e., addressed towards the tip), is physically impossible. The result is for \( \beta > 60^\circ \) the mobilized friction at the cone face will be only a fraction of available one, and the soil will stick the cone \((\psi = 0)\). For instance, for a flat punch \((\beta = 90^\circ, \alpha = 1.0)\), the limit the mobilized friction is about 0.2.

The pressure increment (Fig. 10) decreases with the cone angle, \( \beta \). For rough cones it becomes negative, meaning the presence of suctions near the cone fac...
Fig. 9 $\rho=100$. Exit velocity ratio, $V_c$.

4 CONCLUSIONS

A complete formulation of the elastic-plastic incompressible flow around a cone tip has been presented. The equations have been solved for a wide range of cases.

Three different flow regimes have been identified, in agreement with previous numerical and experimental analyses of cone penetration.

For very sharp cones, ($\beta<5^\circ-10^\circ$), there is an elastic zone ahead the tip, the cone acting by cutting the soil sideways.

The second regime covers the usual range for penetrometers. All the soil is plastic and the tip is a stagnation point.

In both cases, the pressure tends logarithmically to an infinite compression at the tip, with the only exception of the lubricated flat punch ($\beta=90^\circ$, $\alpha=0$) in which it is constant through the whole region.

For rough blunt cones ($\beta>60^\circ$), the soil sticks to the cone face and the cone roughness is only partially mobilized, sometimes producing infinite suctions instead of compressions at the tip.

5 ACKNOWLEDGEMENTS

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REFERENCES


The Piezocone Penetration Test

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In reviewing aspects of the current state of the piezocone test this paper draws principally on the papers on this topic presented at this conference. Other relevant material is included, as well as some of the Author's own views. No attempt is made to review the papers systematically, but instead certain themes which are common to several papers are examined. The importance of using properly defined factors for the interpretation of the cone is emphasised. Several factors derived from the piezocone test depend significantly on the overconsolidation ratio, and estimation of consolidation history from piezocone tests is a promising area of development.

Introduction

1. Seven of the papers at this conference (numbers 14, 15, 17, 18, 19, 23 and 32) are concerned primarily with the piezocone test, and these are the primary source material for this paper; they will be referred to by number, and full references are given at the end of this paper. Papers 28 and 37 also refer to the piezocone. Although included in session 3, paper 16 is not primarily concerned with the piezocone and is not discussed here, whilst papers 23 and 32 can be found in session 4. The second main source of material is the proceedings of the recently held International Symposium on Penetration Testing (ISOPT-1), and in particular the paper to that conference by Wroth on interpretation of cone tests.

2. Three of the above papers (17, 18 and 19) are case histories of particular investigations using the piezocone, whilst the other four are more general research papers concerned with the wider use of the test. Since the application of the pore pressure measurements is primarily of value in clay soils, this paper will concentrate on testing in clays.

Profiling with the Piezocone

3. The first use of the piezocone is as a profiling device, where the piezometric element provides a more sensitive measurement than the tip resistance for detection of thin layers of soil. Used in conjunction with the cone tip measurement the pore pressure reading can give useful information about soil type. In Fig. 5 of Paper 15 by Powell et al., for instance, two thin layers are apparent at about 13m depth in which the tip resistance rises and the pore pressure decreases; these are almost certainly siltier layers in which the tendency to dilate causes lower pore pressures and a higher tip resistance. The detection of such layers can have important consequences, particularly as far as rate of settlement is concerned.

4. Similarly in Fig. 4b of Paper 18 by Wake- ling a more clayey layer is apparent at about 27m where the cone resistance falls and the pore pressure increases. This figure also illustrates a problem in making measurements in dilatant materials where negative excess pore pressures are observed. It can be seen that between 18m and 30m on the profile there is a distinct lower cut-off to the pore pressures at about -80kPa. This is almost certainly due to cavitation in some part of the pore fluid system, and is impossible to avoid even in the highest quality of tests.

Interpretation of the Piezocone

5. In the interpretation of the piezocone test it is useful to draw an analogy with the conventional undrained triaxial compression test. In the triaxial test the initial conditions may be represented by the cell pressure $\sigma_3$ and the pore pressure $u_o$. The equivalent parameters for the piezocone test would be the in situ vertical stress $\sigma_{vo}$ and the pore pressure $u_o$. For the piezocone test it might be more desirable to use either the in situ horizontal stress $\sigma_{ho}$ or mean stress $p_o$, but either of these would require an estimate of $K_s$ to be made, whilst $\sigma_{vo}$ can be estimated solely from $\gamma h$. The in situ pore pressure $u_o$ is estimated either from the position of a static water table, or from other information if that is available, such as the end points of piezocone dissipation tests.
6. At failure in the undrained triaxial test two further measurements are made, the axial stress \( \sigma_1 \) and the pore pressure \( u_f \). Neither of these measurements are useful in themselves, but must be interpreted in terms of changes from the initial conditions. The two useful quantities which can be derived are \( (\sigma_1 - \sigma_3) / \sigma_3 \), the maximum stress difference at failure (the difference between two total stresses) and \( (u_f - u_o) \), the change in pore pressure (the difference between two pore pressures). Finally an effective stress representative of the initial conditions for the test may be obtained as \( (\sigma_3 - u_o) \approx \sigma'_3 \). Note that no total stress or pore pressure is useful in isolation, and only differences between these quantities are useful.

7. Indications of soil type can be obtained from dimensionless ratios of the above quantities. The two most useful are \( (\sigma_1 - \sigma_3) / \sigma'_3 \), which is related to the effective stress strength properties, and is equal to \( 2 \sin \phi' / (1 - \sin \phi' \phi) \), and \( (u_f - u_o) / \sigma_3 - \sigma'_3 \), which is related to the overconsolidation ratio (OCR) and is equal to Skempton's pore pressure parameter at failure \( A_f \).

8. An exactly equivalent procedure should be used for the interpretation of the piezocene test. The measured quantities are the total tip resistance \( q_t \) (after any appropriate corrections for the effects of pore pressure in the groove behind the cone tip), the pore pressure \( u \) and the friction sleeve measurement \( f_s \). The first two are analogous to the quantities \( \sigma_1 \) and \( u_f \) in the triaxial test. The three stress differences which should be derived are therefore \( (q_t - \sigma_{vo}) \), \( (u - u_o) \) and a representative initial effective stress \( \sigma_{vo} / \sigma'_o \). The additional quantities measured, \( f_s \), is a shear stress and may be used directly, but it is worth emphasising that \( q_t \) and \( u \) cannot be used in isolation, nor is the ratio \( u_f / u_o \) meaningful. In Paper 17, Brailsford et al. subtract \( \sigma'_o \) from \( q_t \), a process which the Author considers as incorrect.

9. As with the triaxial test, dimensionless ratios of the above quantities will be indicators of soil type. The analogous ratios are \( (q_t - \sigma_{vo}) / \sigma'_o = Q \) and \( (u - u_o) / (q_t - \sigma_{vo}) = B_q \). In addition the friction ratio should be defined as \( f_s / (q_t - \sigma_{vo}) = F \), and not as in the conventional definition, which is not physically reasonable. Alternative ratios such as \( (u - u_o) / \sigma'_o = QB_q \) and \( f_s / \sigma'_o = QF \) might prove to give clearer indications of soil type, and would be equally acceptable. Although it has not yet been used, the factor \( (q_t - u) / \sigma'_o = Q(1 - B_q) - 1 \), which is analogous to \( \sigma'_1 / \sigma'_3 \) in the triaxial test, may be a useful indicator of soil type, although it must be pointed out that \( (q_t - u) \) is often the small difference between two large quantities. It should, however, be realised that the piezocene test can yield only three independent dimensionless ratios.

10. Since the piezocene imposes much more complex stress conditions on the soil than the triaxial test, the interpretation of the dimensionless ratios is not as simple as for the triaxial test. The simplest analogy appears to be between \( B_q \) and \( A_f \), both of which decrease with increasing overconsolidation ratio. Wroth (1988) shows data from the Onslow site, with \( B_q \) dropping from about 0.7 at an OCR of 1 to about 0.25 at an OCR of 4. Long and O’Riordan (Paper 10, Fig. 5) show the results of tests at nine sites, indicating a reduction of \( B_q \) with OCR but, as might be expected, rather more scatter. Campanella et al. (Paper 23, Fig. 6) present data of \( N_{50} = B_q N_{kt} \) with depth, and then convert this to an implied profile of OCR with depth (Paper 23, Fig. 8). It must be emphasised that correlations between either \( B_q \) or \( N_{50} \) with OCR are sufficiently scattered that this latter plot must be regarded as somewhat tentative.

11. Lutenegger and Kabir (Paper 14, Fig. 5) present the pore pressure results from three sites in the alternative form of \( (u - u_o) / \sigma'_o = QB_q \) against OCR, showing a clear trend of increase from about 4 at an OCR of 1 to about 10 at an OCR of 10. Based on the concepts of cavity expansion theory Mayne and Bachus (1988) suggest a relationship which takes the form \( (u - u_o) / \sigma'_o = 1 + B/OCR^{\lambda} \) where \( B \) is a constant which depends largely on the stiffness to strength ratio \( G/s_u \) for the clay (as well as other factors) and typically may have a value of about 2. The parameter \( \lambda \) is a constant described more fully below in paragraph 12. Mayne and Bachus fit the data from several sites quite successfully with this expression, although there is some considerable scatter to the data.

12. The parameter \( Q \) is also strongly influenced by the overconsolidation ratio. Wroth (1988) again presents the data from the Onslow site in the form of a plot of \( Q \) against OCR, indicating \( Q \) increasing from about 4 at an OCR of 1 to about 13 at an OCR of 4. Since \( Q = N_{kt} \sigma'_o \), the expected shape of this curve may be estimated. The relationship \( \sigma'_o / \sigma_{vo} = A(OCR)^{\lambda} \) is well established empirically as well as being supported by the concepts of critical state soil mechanics. Taking \( A = 0.25 \) and \( \lambda = 0.8 \) (which are reasonable estimates for a typical
clay) and assuming a constant \( N_{kt} \) value of 15 provides a close match to curve given by Wroth. Lutenegger and Kabir (Paper 14, Fig 5) also show \( Q \) values varying from about 6 at an OCR of 1 to 25 or higher at an OCR of 10.

13. Powell et al. collect the data from five U.K. sites in Fig. 10 of Paper 15 to show that the approximate expression \( OCR = K Q \) holds reasonably well at each site, but with \( K \) values varying from 0.2 to 2.2. The very wide range of \( K \) values seems difficult to explain, since the above correlation for \( s_n/\sigma'_{vo} \) is well established with a rather limited range of \( A \) and \( \lambda \) values, and \( N_{kt} \) varies by a factor of only 2 for most of the tests reported (see Fig. 1 later in this paper). The tenfold variation of \( K \) is therefore surprising. It may be that the relevant values of \( A, \lambda \) and \( N_{kt} \) all vary sympathetically for the various clays, but another explanation could lie in the value of the OCR, which can be difficult to determine. This is clearly an area for further work.

14. The factor \( (q_l - u_l)/\sigma'_{vo} = Q(1 - B_l) - 1 \) suggested above would also be a function of overconsolidation ratio, and a simplification of the approach suggested by Konrad and Law (1987) would suggest that this factor would in fact be directly proportional to the OCR, with the constant of proportionality near unity.

15. The final dimensionless ratio is the modified friction ratio \( F \), which is an indicator of soil type, possibly reflecting the effective stress strength properties. Since \( \sigma'_{vo} \) is usually quite small compared to \( q_l \) except for soft clays or for offshore work, the interpretation of the conventional (but badly defined) friction ratio will apply also to the modified ratio for many cases.

Multiple Pore Pressure Measurements

16. The debate about the positioning of the piezometer or the piezcone has not been resolved, and there seems now to be a general consensus that there is no single "best" position for the piezometer. The cone tip is regarded as too vulnerable a position, and measurements half way up the cone face offer almost good profiling ability with less vulnerability. A position just above the cone shoulder is advantageous for making the necessary corrections to the cone tip resistance, and also is useful for the determination of overconsolidation ratio, but has the disadvantage that this appears to be a position where the gradient of pore pressure is very high, and so measurements may be rather prone to error. A measurement further up the cone shaft also has advantages, particularly as the pore pressure gradients will be lower, and the top of the friction sleeve is an obvious position which (in conjunction with a cone shoulder measurement) would allow correction of the friction sleeve measurements for the effects of pore pressures.

17. The pressures at the three most promising points, i.e. half way up the cone face, just behind the cone shoulder and at the top of the friction sleeve will be referred to as \( u_1, u_2 \) and \( u_3 \). In research devices it would be desirable to make all three measurements, but this is unlikely to be adapted in practice. If only one measurement is to be made then it would seem best to offer the option of either \( u_1 \) or \( u_2 \). The cone shoulder measurement \( u_2 \) will be most important in soft clays, where the pore pressure correction to \( q_l \) is relatively large and must be made accurately. In stiff clays \( u_1 \) is likely to be more useful as \( u_2 \) measurements may be rather less repeatable, and may be prone to cavitation problems where negative values are encountered.

18. As discussed above, the initial pore pressure must be subtracted before making use of any pore pressure measurements, and then dimensionless ratios may be of use for indicating soil type. The most useful ratio would appear to be \( (u_2 - u_0)/(u_1 - u_0) = \beta \), or the closely related \( (u_2 - u_1)/(u_1 - u_0) = \beta - 1 \). Some ratios proposed in the literature such as \( u_2/u_1 \) or \( u_2 - u_1)/(u_1 - u_0) \) are not acceptable.

19. Lutenegger and Kabir (Paper 14, Fig. 1) present a schematic diagram indicating only a slight decrease of pore pressure from cone tip to shaft for a normally consolidated clay, and a much more dramatic drop of pressure for an overconsolidated clay. Other data presented by Sills et al. (Paper 32, Fig. 9) show a similar trend, but a significant drop even for a normally consolidated clay. Powell et al. (Paper 15, Fig. 6) present rather similar data, suggesting that at high overconsolidation ratios negative excess pore pressures behind the cone shoulder are observed. However, Wakeling (in verbal discussion to this conference) suggests that such observations of negative excess pore pressures can be somewhat unreliable.

20. Sills et al. (Paper 32, Fig. 7) collect together data on ratios of pore pressures in the form of a plot of \( \beta \) against overconsolidation ratio, showing a distinct trend of reduction of \( \beta \) from about 0.75 at an OCR of 1.3 to about 0.55 at an OCR of 4. Two points from tests in a large laboratory calibration chamber fall somewhat above this trend. Unfortunately the Powell et al. (Paper 15) data cannot be used to extend this plot since they report ratios of total, not excess pore pressures.
Undrained Strength

21. In the triaxial test the undrained strength is related to the stresses by \((\sigma_1 - \sigma_3)/s_u = 2\), and continuing the analogy with the piezocene it is reasonable to adopt the usual definition of the cone factor \((q_t - u)/s_u = N_{kt}\). Again, because the piezocene subjects the soil to a more complex stress regime than in the triaxial test, \(N_{kt}\) is not a constant, but varies according to soil properties. Theoretical research (Housby and Teh, 1988) indicates that \(N_{kt}\) is a function mainly of the stiffness to strength ratio \(G/s_u\) and the relative values of the horizontal and vertical stresses, conveniently quantified as \((\sigma_{vo} - \sigma_{ho})/2s_u\).

22. It should be appreciated that there can be two quite different reasons for using the cone factor \(N_{kt}\). If independent measurements of strength are available then it may be possible to infer information about the clay from the value of \(N_{kt}\). Alternatively if no such information is available then a reliable value of \(N_{kt}\) is required in order to find \(s_u\). Unfortunately this latter process can be unreliable since it is now clear that \(N_{kt}\) is not a constant, but itself depends on the soil properties. This latter process could, however, still be valuable if a reliable site-specific value of the cone factor is available.

23. In defining the \(N_{kt}\) factor it should always be borne in mind that the undrained strength of a clay is not a unique property, but depends on the mode of shearing, and therefore the particular test used to measure it. Furthermore, different tests involve shearing of different volumes of clay, and particularly for stiff fissured clays, this may be important as one test may be measuring the strength of the intact clay and another the strength on fissures or other planes of weakness. There is a popular view that the cone test measures the properties of the soil in a very small zone around the cone, but this view is not supported by the evidence that the cone factor can essentially be explained by cavity expansion theories. The use of such theories indicates that the cone resistance is controlled by the soil properties quite distant (e.g., 10 to 20 radii) from the cone as well as the local properties. This latter view is also supported by the work in large calibration chambers, which show a clear influence of the boundaries even with a chamber radius more than 30 times the cone radius.

24. Long and O’Riordan show in Fig. 4 of Paper 19 a plot indicating \(N_{kt}\) values in London Clay from 13 to 27. It is apparent from their figure that the variation of the cone resistance is much less than the scatter of the strengths measured in the triaxial test. It may be that the cone gives a more repeatable measurement in this stiff clay because it effectively integrates the strength over a large volume, whilst the small triaxial samples are prone to significant variation.

25. Other factors which make use of the undrained strength have also been proposed for the interpretation of the piezocene. The parameter \((u - u_o)/s_u = N_{\Delta\phi}\) is simply equal to \(B_qN_{kt}\) but is probably a less useful quantity than \(B_q\) since it requires an independent measurement of strength as well as the results of the piezocene test. Powell et al. (Paper 15, Fig. 12) make a plot of \(N_{\Delta\phi}\) against \(B_q\), but the groupings of points for the different soils reflect principally a range of \(N_{kt}\) values at different sites from about 25 down to about 12. The same information perhaps can be seen more clearly when the data is replotted as \(N_{kt}\) against \(B_q\) as in Fig. 1 (where the use of \(B_q^*\) indicates that the pore pressure was measured on the cone face). It is tentatively suggested from this plot that there is a trend of reduction of \(N_{kt}\) with increasing \(B_q\).

26. The so-called effective cone factor defined as \((q_t - u)/s_u = N_{ke}\) is much more difficult to justify. It can be shown to be equal to \(N_{kt}(1 - B_q + \frac{1}{2})\), so variations of \(N_{ke}\) therefore principally reflect changes of \(B_q\) and \(Q_1\), both of which depend on the overconsolidation ratio. The \(N_{ke}\) factor is also equal to \(N_{kt}(1 - B_q) + (\sigma_{vo}/s_v)\), and Figure 2 shows a reproduction of Fig. 11 of Paper 15 by Powell et al., in which the added heavy line represents the relationship between \(N_{ke}\) and \(B_q^*\).
$N_{kt} = 15$ and $s_u/\sigma_{vn} = 1/4$. The line is given for example only, but it explains the trend of most of the data. The points plotting below the line at high $B_q$ (i.e. low OCR) indicate a rather lower $N_{kt}$ value and those at low $B_q$ (i.e. high OCR) indicate a combination of a higher $N_{kt}$. The change in $s_u/\sigma_{vn}$ with OCR has little effect since the last term in the expression for $N_{kt}$ is relatively small.

![Figure 2](image.png)

Figure 2, $N_{kt}$ versus $B_q^*$
(after Powell et al.)

27. Two papers (15 and 19) make use of a recently published interpretation method which makes significant use of an “attraction” in interpreting cone tests. It is the Author’s view that the use of this concept is both misleading and erroneous. Similarly, the Author sees no justification in the finite undrained strength at zero vertical effective stress implied by the lines shown on Figs. 10 and 11 in Paper 17 by Braithwaite et al.

**Dissipation Tests**

28. Another application of the piezocone test is to determine consolidation characteristics by carrying out a dissipation test with the cone held static. Lutenegger and Kabir (Paper 14) indicate a difference in the character of the dissipation curve for tests in normally consolidated and overconsolidated soils, with the latter showing an initial rise in pore pressure followed by a fall. This effect is likely to be due to some redistribution of the excess pore pressures around the cone in the early stages of the test, and is likely to be strongly dependent on the position of measurement, occurring only on the shaft of the cone, as shown by Campanella et al. in Fig. 9 of Paper 23. Powell et al. (Paper 15) also give the results of dissipation tests, and although they do not show the initial rise in pore pressure, it is understood that this did occur in the heavily overconsolidated clay and that the dissipation time was measured from the instant of maximum pore pressure.

29. The interpretation of dissipation tests is often carried out using the theory of Baliagh and Lebadoux (1988), but it should be realised that this theory is only for a particular set of soil properties and that dissipation times are strongly influenced by soil stiffness. A similarly based theory, but accounting for soil stiffness, is given by Houlxby and Teh (1988).

30. The end points of dissipation tests can provide valuable information about in situ groundwater conditions, and Wakeling (Paper 18) uses this information to demonstrate that the main hydraulic gradient in a tailings dam is vertically downward.

**Conclusions**

31. By comparison with other soil tests the interpretation of the piezocone is still in its early stages. It is important at this stage that methods of interpretation should be established which are based on sound principles, and in this paper the importance of never using a total stress or pore pressure alone, but only in the form of a stress difference, has been emphasized. Furthermore, the classification of soils may best be achieved by using dimensionless ratios of these stress differences, and following Wroth (1988) the ratios $Q$, $B_q$, and $F$ are favoured for the interpretation of cone data. Robertson presents soil identification charts in discussion at this conference based on these three parameters. For deriving the undrained shear strength the factor $N_{kt}$ should be used. If two piezometers are included on the cone, the ratio $\beta$ is recommended for their interpretation. Whilst other acceptable factors can be derived from combinations of the above five, it is felt to be undesirable to use them unless it can be clearly demonstrated that they help to present a clearer picture of soil behaviour. Any factor which cannot be derived as a function of the five primary factors (e.g. the conventionally defined friction ratio) is not acceptable.

32. Three of the factors, $Q$, $B_q$ and $\beta$ are strong functions of the overconsolidation ratio, and use of these factors, either alone or in combination, together with empirical correlations, seems to promise a useful application of the piezocone in estimating overconsolidation ratios. Whilst the piezocone may have many applications in the future, this appears to be a particular area of interest at present. Several
of the papers in this conference concentrate on this aspect, and together they form an important contribution to the understanding of the piezocene.

References


