Direct Solution of Plasticity Problems in Soils by the Method of Characteristics.

by

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Direct Solution of Plasticity Problems in Soils

by the Method of Characteristics

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SYNOPSIS The method of characteristics is well established as a direct method of solving for the stresses in plastically deforming soils under conditions of plane strain. The application of similar methods to problems of axial symmetry, and to soils with non-homogeneous properties is described and some illustrative examples given. The method of characteristics may also be used for the solution of the plastic displacement equations, although the exact form which these equations should take is still a matter of controversy. Illustrations are given of displacement calculations using the simplifying assumptions either of an associated flow rule (which is unrealistic for a frictional material) or using a fixed rotation term (which, although it has no physical justification, leads to realistic displacement fields).

INTRODUCTION

There are many problems in engineering which reduce to the task of solving an appropriate boundary value problem in continuum mechanics. In soil mechanics for instance the problem of calculating the settlement or the ultimate capacity of a footing, aside from the possibility of using totally empirical methods, amounts to the solution of a boundary value problem. The solution of such a problem involves several stages, first a suitable model for the soil must be chosen and the numerical values of the various parameters describing the model established, then the boundary conditions of the real problem must be simplified into an idealised mathematical form, and finally a solution obtained to the mathematical boundary value problem.

The advent of large digital computers has enormously increased the range of problems which can be solved using this approach, particularly in the areas where closed form algebraic solutions cannot be found to problems, and numerical techniques have to be adopted. The most obvious development is in the field of constitutive relations for soils. Whilst for many problems it may be appropriate to describe the soil either by linear isotropic elasticity or by perfect plasticity, it is now possible to adopt a much more sophisticated constitutive law for the soil where this may prove necessary. The other area where numerical methods are advantageous is the description of problems involving complex geometries, sometimes including several different material types. This is in contrast with most closed form solutions which are restricted to very simple geometries, for instance the solutions for certain simple loads on the isotropic elastic half space.
Prominent in the numerical methods adopted to solve boundary value problems, and indeed so prominent in soil mechanics that it has been used almost to the exclusion of other techniques, has been the Finite Element Method. The method has certain notable advantages: it is well able to accommodate geometrically complex problems, it can handle several material types in a single problem and may be used with complicated constitutive laws. The method does, however, have some disadvantages in its application to non-linear materials, where incremental and/or iterative techniques must be used to satisfy the constitutive laws.

Alternative methods for the direct solution of the equations of plasticity have a long history in soil mechanics. Adopting the Mohr-Coulomb failure criterion as a yield criterion for a rigid-perfectly plastic material then the Coulomb wedge calculation and Rankine’s earth pressure theory may both be seen as direct applications of plasticity theory. In more modern terminology the Rankine calculation would be described as an application of the lower bound theorem of plasticity theory, in which a stress field is found which is in equilibrium and does not violate the yield criterion.

Direct combination of the Mohr-Coulomb failure criterion with the equations of equilibrium leads, as in metal plasticity, to a set of hyperbolic partial differential equations in the stresses which may be solved using the method of characteristics. This method was used extensively by Sokolovskii (1965). Interest in the approach continued into the early 1970's, and for instance Serrano (1972) explored a method for linking the stress and displacement equations in a work hardening theory of plasticity. The development of the method at that stage was reported by Wroth (1972). Since then the method of characteristics has been neglected in favour of the more versatile Finite Element Method, and it is worth examining the reasons why it is appropriate to reconsider the more direct solution methods for plasticity theory at this stage.

There are important theorems of plasticity theory which may be applied to materials for which a model of perfect plasticity with an associated flow rule is applicable. The first is the uniqueness theorem (Drucker, 1959) which determines that for any problem of plastic flow with a particular geometry then a unique solution exists for the loads at plastic collapse. The upper and lower bound theorems then allow this collapse load to be bracketed, and in practice quite close bounds may be found (see for example Davis et al., 1980). For these materials the method of characteristics discussed below leads directly to lower bound solutions for the collapse loads.

For frictional materials an associated flow rule is unrealistic and so the uniqueness theorem no longer applies. The possibility of obtaining non-unique theoretical solutions to a problem is itself an argument in favour of a number of different calculation methods being developed, so that each may be cross-checked with the others. Whilst the bound theorems for a material with a non-associated flow rule are severely weakened (Palmer, 1966) it still seems reasonable that a solution which satisfies the yield and equilibrium conditions may be close to the true solution, and for that reason the method of characteristics is still appropriate.

The second reason for developing direct solutions for the plasticity equations is that the equations of plasticity can lead to discontinuous solutions for either the stresses or the displacements. The Finite Element Method cannot accommodate this type of discontinuous behaviour, which can be described precisely by the method of characteristics. Such discontinuities, particularly of displacement (in the form of slip planes) are well known features of the real behaviour of soils.
In addition to the above reasons the rapid development of the availability of interactive computers with the capacity to solve rapidly the plasticity equations for quite complex problems has generated a new interest in this subject. The speed and power of the method make it ideally suited to parametric studies, and its accuracy allows it to be used to provide yardsticks with which other theoretical methods may be compared.

THE EQUATIONS OF A PLASTICALLY DEFORMING SOIL

Mathematical Form of the Differential Equations

The equations of elasticity, when combined with the equilibrium and compatibility conditions, lead to a set of partial differential equations in the stress increments and strain increments which are elliptical in form. Such a system of elliptical equations may be solved numerically using for instance relaxation techniques. The Finite Element Method is also a highly developed technique for the solution of elliptical equations. The equations of perfect plasticity, when combined with the equilibrium and compatibility conditions, lead, however, to a hyperbolic system of partial differential equations in the stresses and in the strain increments. Hyperbolic equations cannot be solved using relaxation techniques, and the Finite Element Method can only be used to solve such equations by adopting incremental and/or iterative techniques, resulting in much less efficient solutions than for elasticity problems, and numerical results which may be of limited accuracy.

An alternative method for solution of hyperbolic equations in two dimensional problems is the integration of the equations along characteristic directions. The use of the method of characteristics is the subject of this paper, which is directed towards the solution of stress and displacement problems in plastically deforming soils in either plane strain or axial symmetry (both of these cases involving only two spatial coordinates). The directions of the sets of characteristics arise from the mathematical form of the equations, and in this sense the method of characteristics is fundamentally different from the Finite Element Method: the grid of points over which the solution is obtained is not pre-determined arbitrarily, but arises as a natural consequence of the governing equations of the soil. It may be expected therefore that the characteristic directions may have some physical significance, and indeed they coincide with the surfaces on which the stresses satisfy the Mohr-Coulomb failure criterion. These surfaces are often identified as the slip surfaces in the soil on which failure is thought to occur.

The hyperbolic equations of a plastically deforming soil have been stated previously elsewhere, but for the sake of completeness they are repeated here in the most comprehensive form usually used in soil mechanics. Unlike the equations of elasticity, in which the stress increments and strain increments are linked in a single set of equations, the plastic equations fall into two groups. The first set of equations arises from the equilibrium and yield conditions and determine the stresses, and the second set is derived from the flow rule and determine the displacements.

When the equations of equilibrium are combined with the yield condition they lead to a set of hyperbolic differential equations in the stresses. These equations are similar to those used in metal plasticity except for some important and far-reaching complications:
(i) The presence of friction in soil (through the parameter $\phi$) in addition to the cohesion $c$.

(ii) The importance of the self weight $\gamma$ of the soil, which is usually absent or unimportant in metal plasticity problems.

(iii) The presence of spatial variations of $c$ and $\phi$. In particular an increase in cohesion $c$ with depth is a common feature of many soils under undrained conditions.

For all of the above reasons the treatment of the stress equations in soil mechanics is a considerably more difficult subject than in metal plasticity.

The plastic stress equations may be solved for certain problems quite independently of the strain increment equations, and for certain problems (e.g. the calculation of bearing capacity) it may only be necessary to solve the stress equations. These stress equations are therefore first presented in the following section, before consideration of the strain increment or displacement equations.

**Basic Equations of Equilibrium**

In the following discussion the stresses always refer to the total stresses unless otherwise specifically stated. In some problems, however, the pore pressures may be zero, and the following equations for total stresses will also apply to the effective stresses.

Using Cartesian axes as shown in Fig. 1(a), the equilibrium equations for a two dimensional problem in plane strain with the $z$-axis vertically downward may be expressed as:

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0
\]

\[
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = \gamma
\]

and for an axially symmetric problem using the coordinate system shown in Fig. 1(b) the equations may be expressed as:

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{rz}}{\partial z} + \left( \sigma_{rr} - \sigma_{hh} \right)/r = 0
\]

\[
\frac{\partial \tau_{rz}}{\partial r} + \frac{\sigma_{rr}}{r} + \frac{\tau_{rz}}{r} = \gamma
\]

where $\sigma_{hh}$ is the hoop stress. If the $x$-coordinate is substituted for the $r$-coordinate in the axially symmetric case and the following substitutions also made:

\[
\gamma \sin \alpha = 0 \quad \text{for plane strain (i.e. } \alpha=0, \gamma=\gamma_s)\]

\[
\gamma \cos \alpha = \gamma_s
\]

\[
\gamma \sin \alpha = -\left( \sigma_{rr} - \sigma_{hh} \right)/r \quad \text{for axial symmetry}
\]

\[
\gamma \cos \alpha = \gamma_s - \tau_{rz}/r
\]

then the variable $\gamma$ represents an equivalent bulk unit weight at an orientation $\alpha$ which accounts for the effects of the additional terms in the axial symmetry equilibrium equations. Both sets of equilibrium equations may then be expressed as:
\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = \gamma \sin \alpha \tag{7}
\]
\[
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = \gamma \cos \alpha \tag{8}
\]
which correspond exactly to the equations studied by Sokolovskii (1965). Note that if solutions are required for the effective stresses and the pore pressure distribution is known (e.g., from a flow net for a problem involving fully developed seepage) then the definition of effective stress may be introduced into equations (1) and (2) to give:

\[
\frac{\partial \sigma'}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = -\frac{\partial u}{\partial x} \tag{9}
\]
\[
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma'}{\partial z} = \gamma_s - \frac{\partial u}{\partial z} \tag{10}
\]

A similar substitution may be made for the case of axial symmetry. Note that \( u \) is the total, not excess, pore pressure. If the pore pressure corresponds simply to hydrostatic conditions then the sole effect of equations (9) and (10) is to reduce the unit weight from \( \gamma_s \) to the effective unit weight \( \gamma' = \gamma_s - \gamma_w \). The substitution of the terms in \( \gamma \) and \( \sigma \) for the right hand sides of equations (7) and (8) is then straightforward.

In order to solve the equations in the axially symmetric case an assumption must be introduced about the value of the hoop stress \( \sigma_{hh} \). (Note that in the case of plane strain the intermediate principal stress \( \sigma_{ww} \) does not appear in the equilibrium equations, but the hoop stress \( \sigma_{hh} \) appears because the sides of the infinitesimal element for which the equilibrium is studied are not parallel, see Fig. 1(b)). The assumption is made, after Haar and von Karman (1909) that the hoop stress is equal to either the major or the minor principal stress. It can be demonstrated that if an associated flow rule is used in connection with the Tresca yield condition, and the radial displacement is non-zero, then the Haar-von Karman assumption must be satisfied; thus the condition may be seen simply as a natural consequence of the choice of the yield locus and flow rule, and is not an additional arbitrary assumption. The Haar-von Karman hypothesis in fact arises directly from the assumption of any plastic potential with corners as in the Tresca yield locus, and since the Mohr-Coulomb yield criterion is a simple extension of the Tresca yield criterion to frictional behaviour, it is entirely consistent with this assumption to include also the Haar-von Karman condition. In the analysis of each problem it must be determined whether the soil is moving outwards or inwards i.e. whether the hoop stress rate is tensile or compressive. The hoop stress will therefore be taken as equal to the minor principal stress or the major principal stress depending on the sign of the hoop strain.

It will be convenient in the following analysis to introduce two new dependent variables, (shown in Fig. 2 for the case of a purely frictional material) representing a mean stress and the orientation of the major principal stress direction to the z-axis:

\[
\sigma = \frac{(\sigma_{zz} + \sigma_{xx})}{2} \tag{12}
\]
\[
\theta = \tan^{-1}(2\tau_{xx}/(\sigma_{zz} - \sigma_{xx}))^{1/2} \tag{13}
\]

Yield Criterion for Soils

In the following analysis the soil will be considered as rigid-perfectly plastic. For this special case the failure condition is identical with the yield condition, and the Mohr-Coulomb failure criterion will be used to specify
the yield surface. Note that it is only in the special case of perfect plasticity that the yield and failure surfaces are identical. The yield condition may be expressed in the form:

$$(\sigma_{zz} - \sigma_{xx})^2 + 4\tau_{xz}^2 = (2c \cos\phi + (\sigma_{zz} + \sigma_{xx}) \sin\phi)^2$$

(14)

which is a convenient way of expressing the Mohr-Coulomb yield condition:

$$|\tau_n| = c + \sigma_n \tan\phi$$

(15)

This yield condition is automatically satisfied by requiring the stresses to be in the following form:

$$\sigma_{xx} = \sigma - (c \cos\phi + \sigma \sin\phi)\cos2\theta$$

(16)

$$\sigma_{zz} = \sigma + (c \cos\phi + \sigma \sin\phi)\cos2\theta$$

(17)

$$\tau_{xz} = (c \cos\phi + \sigma \sin\phi)\sin2\theta$$

(18)

Then the stresses ($\sigma_{xx}'$, $\sigma_{zz}'$, $\tau_{xz}'$) always satisfy the yield condition.

It has been assumed that the yield condition for the soil has been specified by the Mohr-Coulomb condition. If, however, the yield condition is specified by any envelope $\tau_n = \tau_n(\sigma_n)$ in the Mohr plot, then a similar system of hyperbolic equations is obtained. The details of the form of the equations are given by Salencon (1977) and this method was used for instance by Horne (1976) to study the flow of the so-called "Warren Spring" material which is expressed by an equation of the form:

$$|\tau_n|/c = \left[(\sigma_n + T)/T\right]^n$$

(19)

where $T$ is a tensile strength and $n$ is an exponent describing the curvature of the yield locus. In the most general case then both $c$ and $\phi$ are treated as variables, and upon substituting equations (16) to (18) into (7) and (8) the differentials are given by the equations (20) and (21) shown in Fig. 3. These are two hyperbolic equations in two variables $\sigma$ and $\phi$ and may be treated in a standard manner. The two characteristic directions are at $(\theta \pm \epsilon)$ to the $z$-axis where $\epsilon = \pi/4 - \phi/2$ and are referred to as the $\alpha$ and $\beta$ characteristics respectively.

The solution along these directions:

$$\frac{dx}{dz} = \tan(\theta \pm \epsilon)$$

(22)

is given by:

$$d\sigma \cos\phi \pm 2(c \cos\phi + \sigma \sin\phi)d\theta =$$

$$\left[\gamma \sin(\alpha T\phi) \mp \frac{3c}{\partial z} \cos\phi \mp \frac{\partial}{\partial z}(\sigma \cos\phi - c \sin\phi)\right]dx +$$

$$\left[\gamma \cos(\alpha T\phi) \pm \frac{3c}{\partial x} \cos\phi \pm \frac{\partial}{\partial x}(\sigma \cos\phi - c \sin\phi)\right]dz$$

(23)

The stress equations can be solved subject to known boundary conditions provided that $c$, $\phi$, $\gamma$ and $\alpha$ are known throughout the region for which the solution is required. It is usual to make the assumption that the soil is yielding everywhere, so that $c$ and $\phi$ take their limiting values. Unless a layered soil is being modelled (and little work has been done in this subject) the only usual variation of $c$ or $\phi$ may be an increase of $c$ with depth - usually at a constant
rate. Most real problems in soil mechanics in fact reduce to one of two special cases:

(i) Undrained analysis with $\phi=0$ and $c=c(z)$ for which (23) becomes:
\[
d\sigma + 2c \, d\theta = (\gamma \sin \alpha \pm \frac{3c}{\partial z}) dx + \gamma \cos \alpha \, dz \tag{24}
\]

(ii) Drained analysis with $\gamma=0$ and $\phi$ constant for which (23) becomes:
\[
d\sigma \cos \phi \pm 2\sigma \sin \phi \, d\theta =
\gamma \sin (\alpha \mp \phi) dx + \gamma \cos (\alpha \mp \phi) dz \tag{25}
\]

Boundary Conditions

The boundary conditions on $\sigma$ and $\theta$ may take several forms, and solutions may be obtained to the following types of problems:

(i) $\sigma$ and $\theta$ known on a non-characteristic line AB in Fig. 4(a). The solution may be found throughout the region ABC (The Cauchy problem).

(ii) $\sigma$ and $\theta$ known on two characteristics DE and EF in Fig. 4(b). The solution may be found throughout DEFG (The characteristic problem).

(iii) $\sigma$ and $\theta$ known on one characteristic HI and a range of values of $\sigma$ or $\theta$ given at H, resulting in a degenerate case of (ii) with one characteristic reduced to a point. The solution is obtainable throughout HIJ in Fig. 4(c). (The degenerate problem).

(iv) $\sigma$ and $\theta$ known on one characteristic KL and a relationship between $\sigma$ and $\theta$ known on a non-characteristic line KM. The solution may be found throughout KLM in Fig. 4(d). (The mixed problem).

(v) Certain other combinations of known conditions, e.g. $\sigma$ and $\theta$ known on one characteristic and relations between $\sigma$ and $\theta$ and the coordinates of a curve through one end of the known characteristic (a variation on the mixed problem).

In practice the above regions in which solutions can be obtained are combined in a variety of manners to obtain solutions to real problems. For instance the bearing capacity of a smooth footing (Fig. 4(e)) is obtained by solving a Cauchy problem in NOP starting from the known stress conditions on NO, followed by the degenerate problem in NQ, and finally the mixed problem in NQR to give the required stresses on the surface NR.

In addition to the above forms of boundary value problem, there are some types of problem in which only discontinuous solutions for the stresses are possible. The equilibrium conditions require that the normal and shear stresses on the discontinuity should be continuous, but the normal stress on planes perpendicular to the discontinuity can be discontinuous. Such stress discontinuities may arise from discontinuous boundary conditions, or may be initiated within the body of the soil. The latter condition is particularly important in some problems in axial symmetry. The discontinuities are not introduced arbitrarily, but arise as a natural consequence of the governing differential
equations; it is worth noting that discontinuous solutions are never obtained for elliptical equations, and that the Finite Element Method is incapable of modelling such behaviour. The numerical procedures for calculating the locations of the discontinuities are complex, and result in more general forms of the boundary value problems than those shown in Fig. 4. The details of the calculations involving stress discontinuities are not presented here.

Basic Equations of Plastic Deformation of Soils

In the analysis of plastically deforming materials the ratios between the strain increment components are related to the stress components by a flow rule. Whilst the ratios are fixed, the absolute magnitudes of the strains are not established by the constitutive law for a perfectly plastic material, but are only limited by the boundary conditions. For this reason it is common in plasticity theory to work in terms of strain rates and velocities rather than strain increments and displacements, with the time scale simply being an artificial convention. The ratio between the strain rates is the same as that between the strain increments in a small time interval, and similarly velocities are related to displacement increments. The following discussion is in terms of displacements, which are understood to be small with respect to the dimensions of the problem. The same equations are equally applicable if interpreted in terms of velocities (in the terminology of plasticity theory) although no dynamic behaviour would in fact be implied. The used of displacements is preferred here to avoid any confusion with the important time dependent behaviour of soils due to viscosity, creep and, most importantly, primary consolidation.

In the derivation of the displacement equations in metal plasticity it is assumed that Drucker's stability hypothesis (Drucker, 1951) is obeyed, which results in the identification of the yield locus as the plastic potential. The coaxiality of the direction of major principal strain increment with the direction of major principal stress also follows from the hypothesis for an isotropic yield condition. In the study of frictional materials it is found that the associated flow rule implied by Drucker's hypothesis results in an angle of dilation that is too large; consequently the yield locus cannot serve as a plastic potential. In the absence of the stability postulate, one may also consider the possibility of non-coaxiality. Rather than deriving the strain rates from the plastic potential, it is usual to consider some quasi-physical arguments about the way in which the soil may deform by internal sliding and rotation mechanisms. The exact form of deformation in granular materials is subject to some debate and there is controversy as to how the displacement equations should be expressed. This problem has yet to be resolved in a satisfactory manner, both from the mathematical point of view, and, more importantly, from the point of view of accurately describing experimental results from tests on soils.

Deformation in Plane Strain

The most general form of the displacement equations used for plane strain problems in soil mechanics will therefore be presented. The reduction of these equations by the adoption of various restrictive assumptions is then discussed. Space does not allow the presentation of the complete derivation of the displacement equations, but they arise from the following considerations.

An element of soil is subject to a stress system which satisfies the yield criterion for the soil. On two sets of surfaces the limiting condition is actually satisfied (and these in fact coincide with the stress characteristic directions). It is assumed that the soil deforms by sliding on these two planes, with the intensity of the sliding on the \( \alpha \) and \( \beta \) planes being \( a \) and \( b \)
respectively, see Fig. 5(a) and (b). It is also noted that the spatial orientation of these planes may be itself rotating at a rate \( \Omega_g \) as the stresses change, and further that a rotation at a rate \( \Omega_f \) of the soil element with respect to the stress directions is also possible (Fig. 5(c)). By considering the contributions of each of the three modes of deformation (two slidings and one rotation) with respect to a system of stresses which itself may be rotating, the displacement equations (26) to (29) given in Fig. 6 are obtained for a material which does not dilate. Equation (26) expresses the incompressibility condition, (27) and (28) give the magnitudes of the two rates of simple shear and (29) gives the rotation terms.

Note that the displacement equations cannot be solved independently of the stress equations, since the displacement equations require the solution for \( \theta \) to have been already obtained, where \( \theta \) is the inclination of the major principal stress direction to the z axis (Fig. 1). The rotation term \( \Omega_g \) is the rate of increase of \( \theta \) for a material point:

\[
\Omega_g = \frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial z}
\]  

(30)

where \( u \) and \( v \) are the displacement increments in the x and z directions. The effect of dilation further clouds the quasi-physical arguments which are used to derive the above equations, and results in more complicated expressions (see for example Meherabadi and Cowin (1978) for the case without free rotation).

For the case of dilation the angle of dilation \( \nu \) is introduced which relates the amount of volumetric expansion of the material (\( \hat{\varepsilon} + \dot{\varepsilon} \)) in terms of the principal strain rates in plane strain) to the magnitude \( \dot{\varepsilon}_d \) of the shear strain rate (\( \dot{\varepsilon}_d \)) by:

\[
(\hat{\varepsilon}_1 + \dot{\varepsilon}_3) = (\dot{\varepsilon}_1 - \dot{\varepsilon}_3) \sin \nu
\]  

(31)

a superposed dot being used to indicate differentiation with respect to the time variable. The parameter \( \nu \) may be interpreted geometrically as shown on Fig. 7. The equations (26) to (29) are four partial differential equations in two variables, but involving unknown quantities \( a, b \) and \( (\Omega_g, \Omega_f) \), so some additional assumptions must be made before these equations may be solved.

**Choices of Assumed Mode of Plastic Deformation**

Four different approaches have been made in order to obtain solutions for the displacements from the above equations:

(i) **Coaxiality.** It is sometimes assumed that the principal axes of strain increment and of stress must be coincident for plastically deforming materials. It is found that the special case \( a=b \) corresponds to this condition, and using this with equations (26) to (28) results in two differential equations in the displacements. This approach was made for instance by Davis (1968), who simply assumed coaxiality rather than deriving it from an assumption that \( a=b \). Davis also included the possibility of dilation. For this case the modified form of equation (26) becomes:

\[
\sin 2\theta \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} \right) + \sin \phi \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial x} \right) = 0
\]  

(32)

and equations (27) and (28) are combined to give the coaxiality condition:
\[
\sin 2\theta \frac{\partial u}{\partial z} - \frac{\partial v}{\partial x} + \cos 2\theta \frac{\partial u}{\partial z} + \frac{\partial v}{\partial x} = 0 \tag{33}
\]

The condition of coaxiality is often linked to that of isotropy, for the following reasons. If in a three dimensional material the strain rate depends only on the stress (and not on the stress rate) and the material is isotropic, then it can be shown that coaxiality of the principal directions of stress and of strain rate must occur. The argument depends on the invariance of the constitutive laws under certain transformations of co-ordinates, and if the same argument is applied in two dimensions the assumption of isotropy does not result in the requirement of coaxiality (Mandl and Fernandez Luque, 1970). It may of course be argued that all real materials exist in a three dimensional space, and so if isotropy implies coaxiality in this case then it should also be used for cases when only a two dimensional analysis is used (e.g. for problems of plane strain). In either case the argument fails if the strain rate depends on the stress rate, so the assumption of coaxiality should be distinguished from that of isotropy.

(ii) Single sliding. By assuming only single sliding (a=0 or b=0), equation (26) together with either (27) or (28) gives the required two equations. Although this would seem to be a rather restrictive assumption, it is found that the resulting deformation fields are often physically reasonable. It is used by Mandl and Fernandez Luque (1970), who also included the effects of dilation. This model of course does not give coaxiality of the principal directions of stress and strain rate.

(iii) Determined rotation. By not allowing the free rotation (\(\Omega_z=0\)), equations (26) and (29) may be used, provided that the rotation of the stresses \(\Omega_x\) is known. This is the approach used by Spencer (1964, 1981), and has also been extended to the case where the material dilates by Meherabadi and Cowin (1978). Although the model does not include coaxiality, it is consistent with isotropy since the strain rates depend on the stress rate (through \(\Omega_x\)) as well as the stress. The inclusion of the stress rate in the calculation poses considerable computational problems in all but very simple cases.

(iv) Double sliding, free rotation. If the rotation \(\Omega_x\) is allowed to be unspecified, then equation (29) is of no value. Equations (27) and (28) may, however, be used in the form of inequalities. If the simple shear due to each of the sliding mechanisms is to be in the same sense as the applied shear stress (as surely it must be on thermodynamic grounds if energy is to be dissipated by the deformation), then it follows that \(a>0\) and \(b>0\). These two inequalities may be used with equation (26) to give a range of possible solutions (de Josselin de Jong, 1977). The limits of the range are at \(a=0\) and \(b=0\), the two single sliding mechanisms, and the coaxial case lies within the range. It is found that the angle between the principal directions of stress and of strain rate can be in the range \(\pm\phi/2\). Dilation has been included in the double sliding, free rotating model by Molenkamp (1980).

Whilst this approach can be criticised on the grounds that it does not yield a unique solution to a problem, at least this method errs on the safe side by suggesting a range of behaviour from which one can select a "worst case". Note that whilst the strict thermodynamic requirement of positive energy dissipation requires only that \((a+b)>0\), de Josselin de Jong introduces the stronger requirement that individually \(a>0\) and \(b>0\) on the grounds that the double sliding free rotating mechanism is made up from many infinitesimal single slidings.
Whichever of the four methods above is chosen, the resulting differential equations are hyperbolic, and may also be solved using the method of characteristics. Whereas for a material with an associated flow rule the two families of displacement characteristics are the same as the stress characteristics, this is not necessarily the case for the non-associated material. The orientation of the displacement characteristics depends on the exact form of the assumptions used. In case (i) above the characteristics are at $\pm \pi/4$ to the direction of major principal stress, rather than $\pm (\pi/4 - \phi/2)$. If dilation is included they are at $\pm (\pi/4 - \nu/2)$. In case (ii) one family coincides with one of the families of stress characteristics and the other family is orthogonal to the first. In (iii) the displacement characteristics are coincident with the stress characteristics (and this has been advanced as an argument in favour of this model). In case (iv) the orientation of the displacement characteristics is indeterminate within certain limits. In all but case (ii) the characteristics correspond to the lines of zero extension in the deforming soil.

These orientations are important since in a method which depends on integration along the characteristics it may be that the stresses and displacements cannot be calculated in the same region. If a numerical technique is adopted (as is essential for all problems in axial symmetry) then even within the region for which the calculation of all variables is possible the displacements will be calculated at a different set of data points from the stresses. Elaborate interpolation routines, which may make the method very cumbersome, may have to be employed.

In spite of the uncertainty as to how the displacement equations should be expressed, some progress can be made in obtaining solutions for displacements using the above equations. As the illustrations given below show, the two cases either of an associated flow rule (for the purely cohesive material) or of determined rotation are particularly straightforward.

The equations for the displacement increments on the characteristics may be derived for a soil in plane strain assuming coaxiality as:

$$\sin(\theta \pm \varepsilon^*) \frac{du}{dx} + \cos(\theta \pm \varepsilon^*) \frac{dv}{dx} = 0$$  \hspace{1cm} (34)

where $\varepsilon^* = \pi/4 - \phi/2$ and the orientation of the characteristics is at:

$$\frac{dx}{dz} = \tan(\theta \pm \varepsilon^*)$$  \hspace{1cm} (35)

For the case of the determined value of the rotation the equations on the $\alpha$ and $\beta$ characteristics (identical to the stress characteristics) may be derived as:

$$\cos(\theta \pm \varepsilon) \frac{du}{dx} - \sin(\theta \pm \varepsilon) \frac{dv}{dx} =$$

$$\left(\Omega_\alpha + \Omega_\beta\right) \sin\phi \frac{dx}{\sin(\theta \pm \varepsilon)}$$  \hspace{1cm} (36)

The extension of the displacement equations to axially symmetric problems is not straightforward in that the simple superposition of the sliding mechanisms shown in Fig. 5 is no longer possible. The equations corresponding to the coaxiality condition may, however, be simply derived, and for this case equation (32) becomes:

$$\sin2\theta \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\right] + \sin\phi \left[\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\right] = \frac{u F \sin2\theta}{x}$$  \hspace{1cm} (37)

where $F = (1 \pm \sin \nu)$, the positive sign to be taken when the hoop stress is equal to the major principal stress and the negative sign when it is equal to the minor
principal stress. Equation (34) becomes:

\[ \sin(\theta \pm \epsilon^*) du + \cos(\theta \pm \epsilon^*) dv = \frac{u F dx}{2x \sin(\theta \pm \epsilon^*)} \]  (38)

The determined rotation equations may also be generalised to the axially symmetric case.

EXAMPLES OF CALCULATIONS USING THE METHOD OF CHARACTERISTICS

Stress Calculations for a Frictional Soil

The application of the method of characteristics to obtaining solutions for the stresses in a mass of plastically deforming soil under plane strain conditions is well known, and is comprehensively reported by Sokolovskii (1965), who obtained solutions for both cohesive and frictional soils for problems of footings, slopes and retaining walls (both single and back-to-back). The method has also been reported in applications to footings with strength increasing with depth (Davis and Booker, 1973) as well as a variety of further problems using constant strength parameters.

The application to problems of axial symmetry is less common, although in principle it is no more difficult if a numerical technique is employed. Apart from a study of the smooth circular punch by Cox et al. (1961) and Cox (1962), and a variety of solutions for slopes and footings given by Szczepinski (1974) little work on the axi-symmetric problem has been reported. A problem in axial symmetry will therefore be used as a simple example of the method.

The solution for the stresses on a smooth vertical plane strain retaining wall in a frictional material is a very simple example of the application of the method of characteristics, and for this case reduces to Rankine’s calculation of the coefficient of active earth pressure. The solution for the rough retaining wall is slightly more complex and yields a lower value of the coefficient of active earth pressure (Fig. 8(a) and (b)).

If a similar problem is considered in axial symmetry, (i.e. the ring shaped retaining wall shown in Fig. 9) a calculation can be made for the pressure on a circular retaining wall, as may be used for instance in the construction of a coffer dam. The mesh of characteristics calculated for a smooth wall with a height to radius ratio of 5 is shown in Fig. 8(c). The pressure on the wall now depends on the height to radius ratio, and as the radius is reduced the effect of the arching around the retaining wall becomes more significant and the average coefficient of active earth pressure reduces (Fig. 10). For this particular problem the earth pressure coefficient varies with depth.

Stress Calculations for a Cohesive Soil

Another axially symmetric problem, this time involving passive pressures, is the calculation of the load on a cone indenting the surface of a soil. This problem is clearly of importance in soil mechanics, with applications to a variety of problems. The cone at the surface of a soil (Fig. 11) is relevant to the Liquid Limit test using the fall cone method, since by relating the load on the cone to the strength of the soil it is possible to back analyse the fall cone test to deduce undrained shear strengths. A detailed discussion of this problem is given by Houlsby (1982). The calculation involves determination of the stresses
throughout the mesh of characteristics shown on the right hand side of Fig. 12, and the contours of the stress variable \( \sigma \) non-dimensionalised with respect to the undrained strength \( c_u \), are shown on the left side of Fig. 12 for a 30° cone with a cone adhesion to soil cohesion ratio of 0.5 penetrating a flat region of soil under undrained conditions with a fixed \( c_u \) and ignoring the self weight \( \gamma_s \) of the soil. (The heave around the cone has been ignored for this calculation, but this simplification is thought to have to have only a small effect on the result (Houlshby, 1982)). If the effective stresses are governed by a frictional yield condition, then at a fixed \( c_u \), the averaged effective stress \( (\sigma_{22} - \sigma_{11})/2 = \sigma' \) will be constant at failure. Contours of \( \sigma \) are therefore similar to contours of \( u_z \) (=\( \sigma - \sigma' \)). It is seen therefore that high pore pressures are developed around the tip of the penetrating cone, a result which is entirely to be expected.

Displacement Calculations for a Frictional Soil

The problem of displacement calculations in plane strain may be illustrated by a very simple example, that of a smooth vertical retaining wall in frictional material with a horizontal free surface, failing under passive conditions. The stress solution is straightforward, with vertical and horizontal principal stresses corresponding to Rankine’s solution \( \sigma = \gamma_s z, \sigma = \gamma x (1+\sin \phi)/(1-\sin \phi) \) and the stress characteristics are at \( \pm (\pi/4-\phi/2) \) to the horizontal (illustrated by solid lines in Fig. 13(a) for the case \( \phi = 30^\circ \)). The corresponding displacement solutions may be found for a variety of different assumptions. The displacement boundary condition which will be imposed is that of rotation of the wall about the toe into the soil mass.

If the assumption of an associated flow rule is made then the displacement characteristics simply coincide with the stress characteristics and a displacement field which involves deformation throughout the region for which the stress field is shown in Fig. 13(a) may easily be calculated. The expansion of the material is, however, unrealistic, as is illustrated by the distorted ground profile and surface displacement vectors after 10° rotation shown in Fig. 13(a).

If a non-associated flow rule (e.g. \( \nu = 0 \)) is used then some further assumptions must be made. If the coaxiality condition is introduced then the displacement characteristics are at \( \pm (\pi/4-\nu/2) \) to the horizontal (shown as solid lines in Fig. 13(b), c.f. the dotted lines showing the stress characteristics) and the distorted surface after 10° rotation of the wall is as shown in Fig. 13(b). The network of displacement characteristics for this passive failure problem is not as extensive as the net of stress characteristics, and a question arises as to whether it is appropriate to use the conditions of yielding (necessary for the stress calculation) outside the region for which deformation is calculated.

If a single sliding mechanism is assumed the two possibilities \( a = 0 \) or \( b = 0 \) must be considered. One of these mechanisms \( (b = 0) \) is illustrated in Fig. 13(c), with the displacement characteristics (solid lines) parallel to and orthogonal to the set of stress characteristics on which the slippage occurs. For this mechanism the rotation term \( (\Omega + \Omega_f) \) may be calculated as zero.

Assuming the other single sliding mechanism \( (a = 0) \) results in the displacement field shown in Fig. 13(d), in which both sliding and rotation must be combined to satisfy the boundary condition. Again the sliding occurs on the family of stress characteristics which coincide with displacement characteristics.

The two single sliding mechanisms are clearly the limiting cases of the double sliding free rotating model, which would allow any deformation mechanism between that shown in Figs. 13(c) and (d) (including for example that in Fig. 13(b)).

An intuitive approach to this problem would suggest the mechanism in Fig. 13(c)
as the most likely at failure, and it is worth noting that this mechanism involves the condition \((\Omega_0 + \Omega_t) = 0\). If the zero rotation was assumed then the same solution would be obtained, but this time from equations with the same characteristic directions as the stress equations. This is clearly an advantage for any computational routine since interpolation is avoided.

It is often found, as in the above case, that the zero rotation assumption results in a physically reasonable deformation mechanism, and so this assumption (which has at present no reasonable physically justifiable basis) has been used to obtain predicted failure displacement fields for several problems. Within the context of the double sliding free rotating model this may be seen as obtaining just one of the possible deformation modes (subject to checking the two inequalities \(a > 0\), \(b > 0\)).

Displacement Calculations for a Cohesive Soil

The cone at the surface of a soil may serve as an example of a displacement calculation for a cohesive soil, and the displacement vectors for the cone penetration problem described above are shown in Fig. 14. Note that the largest heave occurs next to the cone, with no movement on the outermost characteristic. The form of the deformed surface due to a continuously penetrating cone may be deduced from this displacement field, with an iterative technique being necessary. The soil surface is first assumed to be flat, a displacement profile is calculated and the deformed surface estimated, the calculation is then repeated starting from the deformed surface and the iteration continued to convergence. Whilst this calculation has been carried out, and a similar method was reported by Lockett (1963) for very blunt cones, the iterative routine is difficult to implement automatically, and a fully satisfactory method has yet to be found.

CONCLUSIONS

The method of characteristics is a direct method of solving the differential equations describing the behaviour of a plastically deforming soil. It is well established for the solution for stress fields in two dimensional problems of plane strain, but may also be applied using numerical techniques for problems in axial symmetry. The partial differential equations for the stresses, accounting for spatial variations in material properties, have been given for a soil with friction and cohesion under conditions of either plane strain or axial symmetry. Illustrations of the use of these equations for solving simple boundary problems for cohesive and frictional soils have been made.

The displacement (or velocity) equations for soils are a subject of some controversy, and a general form of these equations for plane strain has been given. Restrictive assumptions have to be made in the case of frictional materials. The two simplest assumptions to implement numerically are those of either an associated flow rule or a determined value of the rotation; these assumptions both lead to displacement characteristics which are identical to the stress characteristics. The associated flow rule leads to unrealistically high dilation values, but the use of a determined rotation (in the cases studied here taken as zero) leads to realistic displacement fields although it does not have any physical justification. This latter solution corresponds to one possible solution from the range given by de Josselin de Jong's double sliding, free rotating model. Examples have been given of displacement calculations for frictional materials using these assumptions, and for the cohesive material using only the associated flow rule (which is realistic for this special case).
The method of characteristics cannot be used to solve every boundary value problem in plasticity theory, since each type of problem requires a slightly different strategy to calculate the mesh of characteristics, and there are many problems for which the appropriate method has not yet been developed. The method also has the disadvantage that at present it cannot accommodate elastic deformation and is therefore only appropriate for problems of large plastic flow. Within these constraints, however, the method of characteristics is a powerful technique for solving for stresses and displacements in plastically deforming materials, and whilst at present it does not have the generality of the Finite Element Method it can lead to efficient and accurate solutions to many boundary value problems of importance in soil mechanics.

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NOTATION

\( a, b \) sliding intensities on \( \alpha \) and \( \beta \) planes
\( c \) cohesion
\( c_u \) undrained strength
\( T \) tensile strength
\( u, v \) displacements
\( u_w \) pore pressure
\( x, z \) Cartesian coordinates
\( \alpha \) body force direction
\( \gamma \) body force parameter
\( \gamma_s \) bulk unit weight
\( \varepsilon_1, \varepsilon_3 \) principal strains
\( \theta \) orientation of major principal stress
\( \nu \) angle of dilation
\( \sigma \) stress parameter
\( \sigma' \) effective stress
\( \sigma_{bh} \) hoop stress
\( \sigma_{xx}, \sigma_{zz}, T_{xx} \) Cartesian stresses
\( \phi \) angle of internal friction
\( \Omega_\sigma \) material rate of change of \( \theta \)
\( \Omega_f \) free rotation
REFERENCES

Sokolovskii, V.V. (1965) Statics of Granular Media, Pergamon Press, Oxford
Direct Solution of Plasticity Problems in Soils

Fig. 1, Definition of stress components

Fig. 2, Characteristic directions
\[
\frac{\partial \sigma}{\partial x}(1 - \sin \phi \cos 2\theta) + \frac{\partial \sigma}{\partial z} \sin \phi \sin 2\theta + 2 \frac{\partial \theta}{\partial x} (c \cos \phi + \sigma \sin \phi) \sin 2\theta + 2 \frac{\partial \theta}{\partial z} (c \cos \phi + \sigma \sin \phi) \cos 2\theta - \frac{\partial c}{\partial x} \cos \phi \cos 2\theta + \frac{\partial c}{\partial z} \cos \phi \sin 2\theta + \frac{\partial \phi}{\partial x} (c \sin \phi - \sigma \cos \phi) \cos 2\theta - \frac{\partial \phi}{\partial z} (c \sin \phi - \sigma \cos \phi) \sin 2\theta = \gamma \sin \alpha \tag{20}
\]

\[
\frac{\partial \sigma}{\partial x} \sin \phi \sin 2\theta + \frac{\partial \sigma}{\partial z} (1 + \sin \phi \cos 2\theta) + 2 \frac{\partial \theta}{\partial x} (c \cos \phi + \sigma \sin \phi) \cos 2\theta - 2 \frac{\partial \theta}{\partial z} (c \cos \phi + \sigma \sin \phi) \sin 2\theta + \frac{\partial c}{\partial x} \cos \phi \sin 2\theta + \frac{\partial c}{\partial z} \cos \phi \cos 2\theta - \frac{\partial \phi}{\partial x} (c \sin \phi - \sigma \cos \phi) \sin 2\theta - \frac{\partial \phi}{\partial z} (c \sin \phi - \sigma \cos \phi) \cos 2\theta = \gamma \cos \alpha \tag{21}
\]

Fig. 3, Basic stress equations of plastically deforming soil

![Fig. 3, Basic stress equations of plastically deforming soil](image)

Fig. 4, Types of boundary condition

![Fig. 4, Types of boundary condition](image)
Fig. 5, Possible forms of plastic deformation
\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} &= 0 \\
\sin(2\theta+\phi) \left[ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial z} \right] + \cos(2\theta+\phi) \left[ \frac{\partial u}{\partial z} + \frac{\partial v}{\partial x} \right] &= a \sin 2\phi \\
-\sin(2\theta-\phi) \left[ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial z} \right] - \cos(2\theta-\phi) \left[ \frac{\partial u}{\partial z} + \frac{\partial v}{\partial x} \right] &= b \sin 2\phi \\
\sin 2\theta \left[ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial z} \right] + \cos 2\theta \left[ \frac{\partial u}{\partial z} + \frac{\partial v}{\partial x} \right] + \sin \phi \left[ \frac{\partial u}{\partial z} - \frac{\partial v}{\partial x} \right] &= 2 \sin \phi \left[ \Omega_\sigma + \Omega_\xi \right]
\end{align*}
\]

Fig. 6, Displacement equations for soil deforming plastically in plane strain with no volume change

Fig. 7, Definition of angle of dilation
Fig. 9. Ring shaped retaining wall

Fig. 9. Pressures on active retaining walls

(A)

(K_A = 0.333)

(B)

(K_A = 0.262)

(C)

(K_A = 0.146)
Fig. 10, Variation of active pressures with radius of ring-shaped retaining wall for $\phi = 30^\circ$.

Fig. 11, Cone penetration
Fig. 12, Stress characteristics and contours of $\sigma/c_u$ for cone penetration

Fig. 14, Displacement field for cone penetration in a cohesive soil
Fig. 13, Possible displacement fields for a smooth passive retaining wall, $\phi = 30^\circ$. 

(a) 

(b) 

(c) 

(d) 

Direct Solution of Plasticity Problems in Soils